EX 5.2.1: Let vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ such that $\mathbf{v} = (1,2)^T$ and $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Moreover, define inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$ as so: $\langle \mathbf{v}, \mathbf{w} \rangle := v_1 w_1 + v_2 w_2 = \mathbf{v}^T \mathbf{w}$

(a) Compute inner product $\langle \mathbf{v}, \mathbf{w} \rangle$.

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{k=1}^{2} v_k w_k = v_1 w_1 + v_2 w_2 = (1)(3) + (2)(-1) = 3 - 2 = \boxed{1}$$

(b) Are $\mathbf{v}$ & $\mathbf{w}$ orthogonal? $\boxed{\text{No, since inner product } \langle \mathbf{v}, \mathbf{w} \rangle \neq 0}$

(c) Compute induced norms $||\mathbf{v}||$ and $||\mathbf{w}||$.

$$||\mathbf{v}||^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \sum_{k=1}^{2} v_k^2 = v_1^2 + v_2^2 = (1)^2 + (2)^2 = 5 \implies \boxed{||\mathbf{v}|| = \sqrt{5}}$$

$$||\mathbf{w}||^2 = \langle \mathbf{w}, \mathbf{w} \rangle = \sum_{k=1}^{2} w_k^2 = w_1^2 + w_2^2 = (3)^2 + (-1)^2 = 10 \implies \boxed{||\mathbf{w}|| = \sqrt{10}}$$

(d) Compute induced metric $d(\mathbf{v}, \mathbf{w})$.

$$[d(\mathbf{v}, \mathbf{w})]^2 = ||\mathbf{v} - \mathbf{w}||^2 = \left|\left| \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right|\right|^2 = \left|\left| \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right|\right|^2 = \left\langle \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\rangle = 13 \implies \boxed{d(\mathbf{v}, \mathbf{w}) = \sqrt{13}}$$

(e) Compute orthogonal projections $\text{proj}_{\mathbf{w}} \mathbf{v}$ and $\text{proj}_{\mathbf{v}} \mathbf{w}$.

$$\text{proj}_{\mathbf{w}} \mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle} \mathbf{w} = \frac{1}{10} \mathbf{w} = \frac{1}{10}(3, -1)^T = \boxed{\left(\frac{3}{10}, -\frac{1}{10} \right)^T}, \qquad \text{proj}_{\mathbf{v}} \mathbf{w} = \frac{\langle \mathbf{w}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v} = \frac{1}{5} \mathbf{w} = \frac{1}{5}(1, 2)^T = \boxed{\left(\frac{1}{5}, \frac{2}{5} \right)^T}$$

EX 5.2.2: Let matrices $A, B \in \mathbb{R}^{2\times 2}$ such that $A = \begin{bmatrix} 1 & -2 \\ -4 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -3 \\ 0 & -1 \end{bmatrix}$.

Moreover, define inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2\times 2}$ as so: $\langle A, B \rangle := a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$

(a) Compute inner product $\langle A, B \rangle$.

$$\langle A, B \rangle = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} b_{ij} = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22} = (1)(-5) + (-2)(-3) + (-4)(0) + (4)(-1) = \boxed{-3}$$

(b) Are A & B orthogonal? $\boxed{\text{No, since inner product } \langle A, B \rangle \neq 0}$

(c) Compute induced norms $||A||$ and $||B||$.

$$||A||^2 = \langle A, A \rangle = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij}^2 = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 = (1)^2 + (-2)^2 + (-4)^2 + (4)^2 = 37 \implies \boxed{||A|| = \sqrt{37}}$$

$$||B||^2 = \langle B, B \rangle = \sum_{i=1}^{2} \sum_{j=1}^{2} b_{ij}^2 = b_{11}^2 + b_{12}^2 + b_{21}^2 + b_{22}^2 = (-5)^2 + (-3)^2 + (0)^2 + (-1)^2 = 35 \implies \boxed{||B|| = \sqrt{35}}$$

(d) Compute induced metric $d(A, B)$.

$$[d(A, B)]^2 = ||A - B||^2 = \left|\left| \begin{bmatrix} 6 & 1 \\ -4 & 5 \end{bmatrix} \right|\right|^2 = \left\langle \begin{bmatrix} 6 & 1 \\ -4 & 5 \end{bmatrix}, \begin{bmatrix} 6 & 1 \\ -4 & 5 \end{bmatrix} \right\rangle = 78 \implies \boxed{d(A, B) = \sqrt{78}}$$

(e) Compute orthogonal projections $\text{proj}_B A$ and $\text{proj}_A B$.

$$\text{proj}_B A = \frac{\langle A, B \rangle}{\langle B, B \rangle} B = \left(\frac{-3}{35} \right) B = -\frac{3}{35} \begin{bmatrix} -5 & -3 \\ 0 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 3/7 & 9/35 \\ 0 & 3/35 \end{bmatrix}}$$

$$\text{proj}_A B = \frac{\langle B, A \rangle}{\langle A, A \rangle} A = \left(\frac{-3}{37} \right) A = -\frac{3}{37} \begin{bmatrix} 1 & -2 \\ -4 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} -3/37 & 6/37 \\ 12/37 & -12/37 \end{bmatrix}}$$

EX 5.2.3: Let polynomials $p, q \in P_2$ such that $p(t) = t^2 - 3t$ and $q(t) = 4 - t - t^2$.

Moreover, define inner product $\langle \cdot, \cdot \rangle$ on P_2 as so: $\langle p, q \rangle := p(0)q(0) + p(1)q(1) + p(3)q(3)$

(a) Compute inner product $\langle p, q \rangle$.

First, evaluate polynomials p & q at the prescribed points $0, 1, 3$ as these values will be used several times:

$$\begin{aligned} p(0) &= (0)^2 - 3(0) &= 0 \qquad & q(0) = 4 - (0) - (0)^2 &= 4 \\ p(1) &= (1)^2 - 3(1) &= -2 \qquad & q(1) = 4 - (1) - (1)^2 &= 2 \\ p(3) &= (3)^2 - 3(3) &= 0 \qquad & q(3) = 4 - (3) - (3)^2 &= -8 \end{aligned}$$

$\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(3)q(3) = (0)(4) + (-2)(2) + (0)(-8) = \boxed{-4}$

(b) Are $p(t)$ & $q(t)$ orthogonal?

$\boxed{\text{No, since inner product } \langle p, q \rangle \neq 0}$

(c) Compute induced norms $||p||$ and $||q||$.

It's easier & less messy to find the **square norm** first, then take the square root:

$||p||^2 = \langle p, p \rangle = p(0)p(0) + p(1)p(1) + p(3)p(3) = (0)(0) + (-2)(-2) + (0)(0) = 4 \implies \boxed{||p|| = 2}$

$||q||^2 = \langle q, q \rangle = q(0)q(0) + q(1)q(1) + q(3)q(3) = (4)(4) + (2)(2) + (-8)(-8) = 84 \implies \boxed{||q|| = \sqrt{84}}$

(d) Compute induced metric $d(p, q)$.

It's far easier & less work to find the **square metric** first using properties of inner product, then take the square root:

$[d(p, q)]^2 = ||p - q||^2 = \langle p - q, p - q \rangle = \langle p, p \rangle - 2\langle p, q \rangle + \langle q, q \rangle = 4 - 2(-4) + 84 = 96 \implies \boxed{d(p, q) = \sqrt{96}}$

(e) Compute orthogonal projections $\text{proj}_q p$ and $\text{proj}_p q$.

$$\text{proj}_q p = \frac{\langle p, q \rangle}{\langle q, q \rangle} q = \frac{-4}{84} q = -\frac{1}{21}(4 - t - t^2) = \boxed{-\frac{4}{21} + \frac{1}{21} t + \frac{1}{21} t^2}$$

$$\text{proj}_p q = \frac{\langle q, p \rangle}{\langle p, p \rangle} p = \frac{-4}{4} p = (-1)(t^2 - 3t) = \boxed{3t - t^2}$$

EX 5.2.4: Let functions $f, g \in C[0,\pi]$ such that $f(x) = \sin(2x)$ and $g(x) = \cos x$.

Moreover, define inner product $\langle \cdot, \cdot \rangle$ on $C[0,\pi]$ as so: $\langle f, g \rangle := \int_0^\pi f(x)g(x)\, dx$

(a) Compute inner product $\langle f, g \rangle$.

$$\begin{aligned}\langle f, g\rangle &= \int_0^\pi \sin(2x)\cos x\, dx \overset{[*1]}{=} \int_0^\pi (2\sin x \cos x)\cos x\, dx = 2\int_0^\pi \cos^2 x \sin x\, dx \\ &\overset{CV}{=} 2\int_1^{-1} u^2(-du) \overset{[*2]}{=} 2\int_{-1}^1 u^2\, du = 2\left[\frac{1}{3}u^3\right]_{u=-1}^{u=1} \overset{FTC}{=} 2\left[\frac{1}{3}(1)^3 - \frac{1}{3}(-1)^3\right] = 2\left(\frac{2}{3}\right) = \boxed{\frac{4}{3}}\end{aligned}$$

[*1] : $\sin(2\theta) = 2\sin\theta\cos\theta$ (Double-Angle Identity for Sine)

CV: Let $u = \cos x \implies du = -\sin x\, dx \implies \sin x\, dx = -du$ and $u(\pi) = \cos(\pi) = -1$, $u(0) = \cos(0) = 1$

[*2] : $-\int_a^b f(x)\, dx = \int_b^a f(x)\, dx$ (Flip Interval Rule for Definite Integrals)

(b) Are $f(x)$ & $g(x)$ orthogonal?

$\boxed{\text{No, since inner product } \langle f, g\rangle \neq 0}$

(c) Compute induced norms $||f||$ and $||g||$.

It's easier & less messy to find the **square norm** first, then take the square root:

$$\begin{aligned}||f||^2 = \langle f, f\rangle &= \int_0^\pi \sin^2(2x)\, dx \overset{[*3]}{=} \int_0^\pi \frac{1-\cos(4x)}{2}\, dx = \frac{1}{2}\int_0^\pi [1-\cos(4x)]\, dx = \frac{1}{2}\left[x - \frac{1}{4}\sin(4x)\right]_{x=0}^{x=\pi} \\ &\overset{FTC}{=} \frac{1}{2}\left[\left(\pi - \frac{1}{4}\sin(4\pi)\right) - \left(0 - \frac{1}{4}\sin(0)\right)\right] = \frac{\pi}{2} \implies \boxed{||f|| = \sqrt{\frac{\pi}{2}}}\end{aligned}$$

$$\begin{aligned}||g||^2 = \langle g, g\rangle &= \int_0^\pi \cos^2 x\, dx \overset{[*4]}{=} \int_0^\pi \frac{1+\cos(2x)}{2}\, dx = \frac{1}{2}\int_0^\pi [1+\cos(2x)]\, dx = \frac{1}{2}\left[x + \frac{1}{2}\sin(2x)\right]_{x=0}^{x=\pi} \\ &\overset{FTC}{=} \frac{1}{2}\left[\left(\pi + \frac{1}{2}\sin(2\pi)\right) - \left(0 + \frac{1}{2}\sin(0)\right)\right] = \frac{\pi}{2} \implies \boxed{||g|| = \sqrt{\frac{\pi}{2}}}\end{aligned}$$

[*3] : $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$ (Half-Angle Identity for Sine)

[*4] : $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ (Half-Angle Identity for Cosine)

(d) Compute induced metric $d(f, g)$.

It's far easier & less work to find the **square metric** first using properties of inner product, then take the square root:

$$[d(f,g)]^2 = ||f-g||^2 = \langle f-g, f-g\rangle = \langle f, f\rangle - 2\langle f, g\rangle + \langle g, g\rangle = \frac{\pi}{2} - 2\left(\frac{4}{3}\right) + \frac{\pi}{2} = \pi - \frac{8}{3} \implies \boxed{d(f,g) = \sqrt{\pi - \frac{8}{3}}}$$

Notice that this was far less work than directly computing the integral $[d(f,g)]^2 = ||f-g||^2 = \int_0^\pi [\sin(2x) - \cos x]^2\, dx$

Here's why "foiling" the inner product $\langle f-g, f-g\rangle$ works (using axioms & properties of inner product spaces):

$$\langle f-g, f-g\rangle \overset{IPS5}{=} \langle f-g, f\rangle - \langle f-g, g\rangle \overset{IP3}{=} (\langle f, f\rangle - \langle g, f\rangle) - (\langle f, g\rangle - \langle g, g\rangle) \overset{IPS3}{=} \langle f, f\rangle - \langle f, g\rangle - \langle f, g\rangle + \langle g, g\rangle = \langle f, f\rangle - 2\langle f, g\rangle + \langle g, g\rangle$$

(e) Compute orthogonal projections $\text{proj}_g f$ and $\text{proj}_f g$.

$$\text{proj}_g f = \frac{\langle f, g\rangle}{\langle g, g\rangle} g = \frac{4/3}{\pi/2}\cos x = \boxed{\frac{8}{3\pi}\cos x}$$

$$\text{proj}_f g = \frac{\langle g, f\rangle}{\langle f, f\rangle} f = \frac{4/3}{\pi/2}\sin(2x) = \boxed{\frac{8}{3\pi}\sin(2x)}$$