INNER PRODUCT SPACES [LARSON 5.2]

- **INNER PRODUCT (DEFINITION):** Let V be a vector space. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha \in \mathbb{R}$.

  An **inner product** on V is a function $\langle \cdot, \cdot \rangle : V \to \mathbb{R}$ satisfying the following inner product axioms:

(IPS1)	$\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$	Non-negativity of Self-Inner Product
(IPS2)	$\langle \mathbf{u}, \mathbf{u} \rangle = 0 \iff \mathbf{u} = \vec{\mathbf{0}}$	Only $\vec{\mathbf{0}}$ has Self-Inner Product of Zero
(IPS3)	$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$	Commutativity of Inner Product
(IPS4)	$\langle \alpha\mathbf{u}, \mathbf{v} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle$	Inner Product of SM is SM of Inner Product
(IPS5)	$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$	Distributivity of VA over Inner Product

- **INNER PRODUCT SPACE (DEFINITION):**

  A vector space V with an inner product $\langle \cdot, \cdot \rangle$ is called an **inner product space**.

  A compact notation for an inner product space is: $(V, \langle \cdot, \cdot \rangle)$

- **INNER PRODUCT (PROPERTIES):** Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha \in \mathbb{R}$.

(IP1)	$\langle \mathbf{u}, \alpha\mathbf{v} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle$	Associativity of Inner Product
(IP2)	$\langle \mathbf{u}, \vec{\mathbf{0}} \rangle = \langle \vec{\mathbf{0}}, \mathbf{u} \rangle = 0$	Inner Product with $\vec{\mathbf{0}}$ is Zero Scalar
(IP3)	$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$	Distributivity of Inner Product over VA

- **THE NORM INDUCED BY AN INNER PRODUCT:** Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Let $\mathbf{u} \in V$.

  The **norm** $||\cdot||$ **induced by the inner product** $\langle \cdot, \cdot \rangle$ **of** V is defined to be: $||\mathbf{u}|| := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$

- **NORMED VECTOR SPACE (DEFINITION):** A vector space V with norm $||\cdot||$ is called **normed vector space**.

  A compact notation for a normed vector space is: $(V, ||\cdot||)$

- **THE METRIC INDUCED BY A NORM:** Let $(V, ||\cdot||)$ be a normed vector space. Let $\mathbf{u}, \mathbf{v} \in V$.

  The **metric** $d(\cdot, \cdot)$ **induced by the norm** $||\cdot||$ **of** V is defined to be: $d(\mathbf{u}, \mathbf{v}) := ||\mathbf{u} - \mathbf{v}||$

- **INNER PRODUCT (ORTHOGONALITY):** "Vectors" $\mathbf{v}, \mathbf{w}$ are **orthogonal** $\iff \mathbf{v} \perp \mathbf{w} \iff \langle \mathbf{v}, \mathbf{w} \rangle = 0$

- **(ORTHOGONAL) PROJECTION ONTO A "VECTOR":** $\text{proj}_{\mathbf{w}}\mathbf{v} := \left(\dfrac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle} \right) \mathbf{w}$

- **STANDARD INNER PRODUCT SPACES:**

INNER PRODUCT SPACE	**PROTOTYPE "VECTORS"**	**INNER PRODUCT** $\langle \cdot, \cdot \rangle$	**INDUCED NORM** $\|\|\cdot\|\|$	**INDUCED METRIC** $d(\cdot, \cdot)$
$\mathbb{R}^n$	$\mathbf{u} = (u_1, \ldots, u_n)^T, \ \mathbf{v} = (v_1, \ldots, v_n)^T$	$\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{k=1}^{n} u_k v_k$	$\|\|\mathbf{u}\|\| := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$	$d(\mathbf{u}, \mathbf{v}) := \|\|\mathbf{u} - \mathbf{v}\|\|$
$\mathbb{R}^{m \times n}$	$A = [a_{ij}]_{m \times n}, \ B = [b_{ij}]_{m \times n}$	$\langle A, B \rangle := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$	$\|\|A\|\| := \sqrt{\langle A, A \rangle}$	$d(A, B) := \|\|A - B\|\|$
P_n	$p(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n$ $q(t) = q_0 + q_1 t + q_2 t^2 + \cdots + q_n t^n$ scalars $t_1, \ldots, t_{n+1} \in \mathbb{R}$	$\langle p, q \rangle := \sum_{k=1}^{n+1} p(t_k) q(t_k)$	$\|\|p\|\| := \sqrt{\langle p, p \rangle}$	$d(p, q) := \|\|p - q\|\|$
$C[a, b]$	$f(x), \ g(x)$	$\langle f, g \rangle := \int_a^b f(x) g(x) \ dx$	$\|\|f\|\| := \sqrt{\langle f, f \rangle}$	$d(f, g) := \|\|f - g\|\|$

REMARK: Other inner products are possible with these spaces but such inner products won't be considered here.

EX 5.2.1: Let vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ such that $\mathbf{v} = (1,2)^T$ and $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Moreover, define inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$ as so: $\langle \mathbf{v}, \mathbf{w} \rangle := v_1 w_1 + v_2 w_2 = \mathbf{v}^T \mathbf{w}$

(a) Compute inner product $\langle \mathbf{v}, \mathbf{w} \rangle$.

(b) Are $\mathbf{v}$ & $\mathbf{w}$ orthogonal?

(c) Compute norms $||\mathbf{v}||$ and $||\mathbf{w}||$.

(d) Compute metric $d(\mathbf{v}, \mathbf{w})$.

(e) Compute orthogonal projections $\text{proj}_{\mathbf{w}} \mathbf{v}$ and $\text{proj}_{\mathbf{v}} \mathbf{w}$.

EX 5.2.2: Let matrices $A, B \in \mathbb{R}^{2\times 2}$ such that $A = \begin{bmatrix} 1 & -2 \\ -4 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -3 \\ 0 & -1 \end{bmatrix}$.

Moreover, define inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2\times 2}$ as so: $\langle A, B \rangle := a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$

(a) Compute inner product $\langle A, B \rangle$.

(b) Are A & B orthogonal?

(c) Compute norms $||A||$ and $||B||$.

(d) Compute metric $d(A, B)$.

(e) Compute orthogonal projections $\text{proj}_B A$ and $\text{proj}_A B$.

EX 5.2.3: Let polynomials $p, q \in P_2$ such that $p(t) = t^2 - 3t$ and $q(t) = 4 - t - t^2$.
Moreover, define inner product $\langle \cdot, \cdot \rangle$ on P_2 as so: $\langle p, q \rangle := p(0)q(0) + p(1)q(1) + p(3)q(3)$

(a) Compute inner product $\langle p, q \rangle$.

(b) Are $p(t)$ & $q(t)$ orthogonal?

(c) Compute norms $||p||$ and $||q||$.

(d) Compute metric $d(p, q)$.

(e) Compute orthogonal projections $\text{proj}_q p$ and $\text{proj}_p q$.

EX 5.2.4: Let functions $f, g \in C[0,\pi]$ such that $f(x) = \sin(2x)$ and $g(x) = \cos x$.

Moreover, define inner product $\langle \cdot, \cdot \rangle$ on $C[0,\pi]$ as so: $\langle f, g \rangle := \int_0^{\pi} f(x)g(x)\, dx$

(a) Compute inner product $\langle f, g \rangle$.

(b) Are $f(x)$ & $g(x)$ orthogonal?

(c) Compute norms $||f||$ and $||g||$.

(d) Compute metric $d(f, g)$.

(e) Compute orthogonal projections $\text{proj}_g f$ and $\text{proj}_f g$.