EX 5.3.3: Let vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $\mathbf{u} = (4, -3, 0)^T$, $\mathbf{v} = (1, 2, 0)^T$, and $\mathbf{w} = (0, 0, 4)^T$.
Moreover, define inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$ as so: $\langle \mathbf{v}, \mathbf{w} \rangle := v_1 w_1 + v_2 w_2 + v_3 w_3 = \mathbf{v}^T \mathbf{w}$

Perform the classical Gram-Schmidt process to produce orthonormal basis $\widehat{\mathcal{Q}} := \{\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2, \widehat{\mathbf{q}}_3\}$ for span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

First of all, label the given vectors with **indexing** to make stating Classical Gram-Schmidt straightforward:

$$\text{Let } \mathbf{v}_1 := \mathbf{u}, \quad \mathbf{v}_2 := \mathbf{v}, \quad \mathbf{v}_3 := \mathbf{w}$$

Now perform Classical Gram-Schmidt orthonormalization: $\Big($Here: $\mathcal{Q}_1 := \{\mathbf{q}_1\}$, $\mathcal{Q}_2 := \{\mathbf{q}_1, \mathbf{q}_2\}\Big)$

$$\begin{aligned}
\mathbf{q}_1 &= \mathbf{v}_1 &&= \mathbf{v}_1 &&= \mathbf{v}_1 \\
\mathbf{q}_2 &= \mathbf{v}_2 - \text{proj}_{span(\mathcal{Q}_1)}\mathbf{v}_2 &&= \mathbf{v}_2 - \text{proj}_{\mathbf{q}_1}\mathbf{v}_2 &&= \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{q}_1 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle}\mathbf{q}_1 \\
\mathbf{q}_3 &= \mathbf{v}_3 - \text{proj}_{span(\mathcal{Q}_2)}\mathbf{v}_3 &&= \mathbf{v}_3 - \text{proj}_{\mathbf{q}_1}\mathbf{v}_3 - \text{proj}_{\mathbf{q}_2}\mathbf{v}_3 &&= \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{q}_1 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle}\mathbf{q}_1 - \frac{\langle \mathbf{v}_3, \mathbf{q}_2 \rangle}{\langle \mathbf{q}_2, \mathbf{q}_2 \rangle}\mathbf{q}_2
\end{aligned}$$

$$\Longrightarrow \mathbf{q}_1 = \mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$$

$$\langle \mathbf{v}_2, \mathbf{q}_1 \rangle = \left\langle \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\rangle = (1)(4) + (2)(-3) + (0)(0) = -2$$

$$\langle \mathbf{q}_1, \mathbf{q}_1 \rangle = \left\langle \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\rangle = (4)(4) + (-3)(-3) + (0)(0) = 25$$

$$\langle \mathbf{v}_3, \mathbf{q}_1 \rangle = \left\langle \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\rangle = (0)(4) + (0)(-3) + (4)(0) = 0$$

$$\Longrightarrow \mathbf{q}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{q}_1 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle}\mathbf{q}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{(-2)}{25}\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 8/25 \\ -6/25 \\ 0 \end{bmatrix} = \begin{bmatrix} 33/25 \\ 44/25 \\ 0 \end{bmatrix}$$

$$\langle \mathbf{v}_3, \mathbf{q}_2 \rangle = \left\langle \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 33/25 \\ 44/25 \\ 0 \end{bmatrix} \right\rangle = (0)\left(\frac{33}{25}\right) + (0)\left(\frac{44}{25}\right) + (4)(0) = 0$$

$$\langle \mathbf{q}_2, \mathbf{q}_2 \rangle = \left\langle \begin{bmatrix} 33/25 \\ 44/25 \\ 0 \end{bmatrix}, \begin{bmatrix} 33/25 \\ 44/25 \\ 0 \end{bmatrix} \right\rangle = \left(\frac{33}{25}\right)\left(\frac{33}{25}\right) + \left(\frac{44}{25}\right)\left(\frac{44}{25}\right) + (0)(0) = \frac{33^2 + 44^2}{25^2}$$

$$\Longrightarrow \mathbf{q}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{q}_1 \rangle}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle}\mathbf{q}_1 - \frac{\langle \mathbf{v}_3, \mathbf{q}_2 \rangle}{\langle \mathbf{q}_2, \mathbf{q}_2 \rangle}\mathbf{q}_2 = \mathbf{v}_3 - \frac{(0)}{\langle \mathbf{q}_1, \mathbf{q}_1 \rangle}\mathbf{q}_1 - \frac{(0)}{\langle \mathbf{q}_2, \mathbf{q}_2 \rangle}\mathbf{q}_2 = \mathbf{v}_3 - \vec{\mathbf{0}} - \vec{\mathbf{0}} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$||\mathbf{q}_1|| = \sqrt{4^2 + (-3)^2 + 0^2} = 5, \quad ||\mathbf{q}_2|| = \sqrt{\left(\frac{33}{25}\right)^2 + \left(\frac{44}{25}\right)^2 + 0^2} = \sqrt{\frac{11^2(3^2 + 4^2)}{25^2}} = \frac{11}{5}, \quad ||\mathbf{q}_3|| = \sqrt{0^2 + 0^2 + 4^2} = 4$$

$$\widehat{\mathbf{q}}_1 = \frac{\mathbf{q}_1}{||\mathbf{q}_1||} = \frac{1}{5}\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -3/5 \\ 0 \end{bmatrix}, \quad \widehat{\mathbf{q}}_2 = \frac{\mathbf{q}_2}{||\mathbf{q}_2||} = \frac{5}{11}\begin{bmatrix} 33/25 \\ 44/25 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix}, \quad \widehat{\mathbf{q}}_3 = \frac{\mathbf{q}_3}{||\mathbf{q}_3||} = \frac{1}{4}\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad \boxed{\text{Orthonormal Basis } \widehat{\mathcal{Q}} = \{\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2, \widehat{\mathbf{q}}_3\} = \left\{ \begin{bmatrix} 4/5 \\ -3/5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

EX 5.3.4: Let functions $f, g, h \in C[0,\pi]$ such that $f(x) = 4,\ \ g(x) = x,\ $ and $\ h(x) = \sin x$.

Moreover, define inner product $\langle \cdot, \cdot \rangle$ on $C[0,1]$ as so: $\quad \langle f, g \rangle := \int_0^\pi f(x)g(x)\ dx$

Perform the classical Gram-Schmidt process to determine an orthonormal basis $\widehat{\mathcal{Q}} := \{\widehat{q}_1, \widehat{q}_2, \widehat{q}_3\}$ for span$\{f, g, h\}$.

First of all, label the given vectors with **indexing** to make stating Classical Gram-Schmidt straightforward:

$$\text{Let } \ v_1(x) := f(x), \quad v_2(x) := g(x), \quad v_3(x) := h(x)$$

Now perform Classical Gram-Schmidt orthonormalization: $\Big(\text{Here: } \ \mathcal{Q}_1 := \{q_1\}, \ \ \mathcal{Q}_2 := \{q_1, q_2\}\Big)$

$$\begin{aligned}
q_1(x) &= v_1(x) &&= v_1(x) &&= v_1(x) \\
q_2(x) &= v_2(x) - \text{proj}_{span(\mathcal{Q}_1)} v_2(x) &&= v_2(x) - \text{proj}_{q_1} v_2(x) &&= v_2(x) - \frac{\langle v_2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1(x) \\
q_3(x) &= v_3(x) - \text{proj}_{span(\mathcal{Q}_2)} v_3(x) &&= v_3(x) - \text{proj}_{q_1} v_3(x) - \text{proj}_{q_2} v_3(x) &&= v_3(x) - \frac{\langle v_3, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1(x) - \frac{\langle v_3, q_2 \rangle}{\langle q_2, q_2 \rangle} q_2(x)
\end{aligned}$$

$$\langle v_2, q_1 \rangle = \langle x, 4 \rangle = \int_0^\pi 4x\ dx = \Big[2x^2\Big]_{x=0}^{x=\pi} \overset{FTC}{=} 2(\pi)^2 - 2(0)^2 = 2\pi^2$$

$$\langle q_1, q_1 \rangle = \langle 4, 4 \rangle = \int_0^\pi 16\ dx = \Big[16x\Big]_{x=0}^{x=\pi} \overset{FTC}{=} 16(\pi) - 16(0) = 16\pi$$

$$\langle v_3, q_1 \rangle = \langle \sin x, 4 \rangle = \int_0^\pi 4 \sin x\ dx = \Big[-4\cos x\Big]_{x=0}^{x=\pi} \overset{FTC}{=} -4\cos\pi - (-4\cos 0) = 4 + 4 = 8$$

$$\langle v_3, q_2 \rangle = \Big\langle \sin x, x - \frac{\pi}{2} \Big\rangle = \int_0^\pi \left(x - \frac{\pi}{2}\right) \sin x\ dx \overset{IBP}{=} \left[\left(\frac{\pi}{2} - x\right)\cos x\right]_{x=0}^{x=\pi} + \int_0^\pi \cos x\ dx = 0 + 0 = 0$$

$$\langle q_2, q_2 \rangle = \Big\langle x - \frac{\pi}{2}, x - \frac{\pi}{2} \Big\rangle = \int_0^\pi \left(x - \frac{\pi}{2}\right)^2 dx = \int_0^\pi \left(x^2 - \pi x + \frac{\pi^2}{4}\right) dx = \left[\frac{1}{3}x^3 - \frac{\pi}{2}x^2 + \frac{\pi^2}{4}x\right]_{x=0}^{x=\pi} \overset{FTC}{=} \frac{\pi^3}{12}$$

$$\implies q_1(x) = v_1(x) = 4$$

$$\implies q_2(x) = v_2(x) - \frac{\langle v_2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1(x) = x - \left(\frac{2\pi^2}{16\pi}\right)(4) = x - \frac{\pi}{2}$$

$$\implies q_3(x) = v_3(x) - \frac{\langle v_3, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1(x) - \frac{\langle v_3, q_2 \rangle}{\langle q_2, q_2 \rangle} q_2(x) = \sin x - \frac{8}{16\pi}(4) - \frac{(0)}{\langle q_2, q_2 \rangle} q_2(x) = \sin x - \frac{2}{\pi} - 0 = \sin x - \frac{2}{\pi}$$

$$||q_1|| = \sqrt{\langle q_1, q_1 \rangle} = \sqrt{16\pi} = 4\sqrt{\pi}, \qquad ||q_2|| = \sqrt{\langle q_2, q_2 \rangle} = \sqrt{\frac{\pi^3}{12}} = \frac{\pi^{3/2}}{\sqrt{12}}$$

$$||q_3||^2 = \langle q_3, q_3 \rangle = \int_0^\pi \left(\sin x - \frac{2}{\pi}\right)^2 dx = \int_0^\pi \left(\sin^2 x - \frac{4}{\pi}\sin x + \frac{4}{\pi^2}\right) dx = \frac{\pi}{2} - \frac{4}{\pi} = \frac{\pi^2 - 8}{2\pi} \implies ||q_3|| = \sqrt{\frac{\pi^2 - 8}{2\pi}}$$

$$\widehat{q}_1(x) = \frac{q_1}{||q_1||} = \left(\frac{1}{4\sqrt{\pi}}\right)(4) = \frac{1}{\sqrt{\pi}}, \qquad \widehat{q}_2(x) = \frac{q_2}{||q_2||} = \left(\frac{\sqrt{12}}{\pi^{3/2}}\right)\left(x - \frac{\pi}{2}\right), \qquad \widehat{q}_3(x) = \frac{q_3}{||q_3||} = \sqrt{\frac{2\pi}{\pi^2 - 8}}\left(\sin x - \frac{2}{\pi}\right)$$

$$\therefore \quad \boxed{\text{Orthonormal Basis } \widehat{\mathcal{Q}} = \{\widehat{q}_1, \widehat{q}_2, \widehat{q}_3\} = \left\{\frac{1}{\sqrt{\pi}}, \left(\frac{\sqrt{12}}{\pi^{3/2}}\right)\left(x - \frac{\pi}{2}\right), \sqrt{\frac{2\pi}{\pi^2 - 8}}\left(\sin x - \frac{2}{\pi}\right)\right\}}$$