ORTHONORMAL BASES, SUBSPACE PROJECTIONS [LARSON 5.3]

- **ORTHOGONAL SET (DEFINITION):** Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space.

  Then set $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ is an **orthogonal set** if:

  $$\langle \mathbf{q}_i, \mathbf{q}_j \rangle = 0 \quad \text{for } i \neq j$$

  i.e. Every distinct pair of basis vectors in $\mathcal{Q}$ is orthogonal.

- **ORTHOGONAL SETS ARE LINEARLY INDEPENDENT COROLLARY – OSLIC:**

  Given inner product space $(V, \langle \cdot, \cdot \rangle)$ and orthogonal set $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$.

  Then, the orthogonal set is linearly independent.

- **ORTHOGONAL BASIS (DEFINITION):** Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space.

  Then basis $\mathcal{Q} = \{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ is an **orthogonal basis** for V if:

  $$\langle \mathbf{q}_i, \mathbf{q}_j \rangle = 0 \quad \text{for } i \neq j$$

  i.e. Every distinct pair of basis vectors in $\mathcal{Q}$ is orthogonal.

- **ORTHONORMAL BASIS (DEFINITION):** Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space.

  Then basis $\widehat{\mathcal{Q}} = \{\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2, \ldots, \widehat{\mathbf{q}}_n\}$ is an **orthonormal basis** for V if:

  $$\langle \widehat{\mathbf{q}}_i, \widehat{\mathbf{q}}_j \rangle = \delta_{ij} \iff \begin{cases} \langle \widehat{\mathbf{q}}_i, \widehat{\mathbf{q}}_j \rangle = 0 & \text{if } i \neq j \\ \langle \widehat{\mathbf{q}}_i, \widehat{\mathbf{q}}_j \rangle = 1 & \text{if } i = j \end{cases}$$

  i.e. Every distinct pair of unit basis vectors in $\widehat{\mathcal{Q}}$ is orthogonal.

- **(ORTHOGONAL) PROJECTION ONTO A SUBSPACE (DEFINITION):**

  Let $\mathcal{Q} = \{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_p\}$ be an orthogonal basis for subspace W of $\mathbb{R}^n$. $(p \leq n)$

  Then the **(orthogonal) projection of vector $\mathbf{v} \in \mathbb{R}^n$ onto subspace W** is:

  $$\text{proj}_W \mathbf{v} := \text{proj}_{span(\mathcal{Q})} \mathbf{v} = \text{proj}_{\mathbf{q_1}} \mathbf{v} + \text{proj}_{\mathbf{q_2}} \mathbf{v} + \cdots + \text{proj}_{\mathbf{q}_p} \mathbf{v}$$

- **CONVERTING $[\mathbf{x}]_{\mathcal{E}} \to [\mathbf{x}]_{\widehat{\mathcal{Q}}}$ (PROCEDURE):** Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space.

  Let $\widehat{\mathcal{Q}} = \{\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2, \ldots, \widehat{\mathbf{q}}_n\}$ be an ordered orthonormal basis for V.

  Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the ordered standard basis for V.

  GIVEN: Vector $\mathbf{x} \in V$ in standard basis coordinates: $\mathbf{x} = [\mathbf{x}]_{\mathcal{E}}$

  TASK: Write vector $\mathbf{x}$ in non-std orthonormal basis $\widehat{\mathcal{Q}}$-coordinates: $[\mathbf{x}]_{\widehat{\mathcal{Q}}}$

  $$(1) \quad [\mathbf{x}]_{\widehat{\mathcal{Q}}} = \text{proj}_{span(\widehat{\mathcal{Q}})} [\mathbf{x}]_{\mathcal{E}} = \text{proj}_{\widehat{\mathbf{q}}_1} \mathbf{x} + \text{proj}_{\widehat{\mathbf{q}}_2} \mathbf{x} + \cdots + \text{proj}_{\widehat{\mathbf{q}}_n} \mathbf{x} = \begin{bmatrix} \langle \mathbf{x}, \widehat{\mathbf{q}}_1 \rangle \\ \langle \mathbf{x}, \widehat{\mathbf{q}}_2 \rangle \\ \vdots \\ \langle \mathbf{x}, \widehat{\mathbf{q}}_n \rangle \end{bmatrix}$$

GRAM-SCHMIDT ORTHONORMALIZATION [LARSON 5.3]

- **(CLASSICAL) GRAM-SCHMIDT ORTHONORMALIZATION, WITH LATE NORMALIZATION:**

  GIVEN: Induced-norm inner product space $(V, \langle\cdot,\cdot\rangle, ||\cdot||)$ & Basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$.

  TASK: Find an orthonormal basis $\widehat{\mathcal{Q}} = \{\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2, \ldots, \widehat{\mathbf{q}}_n\}$ for span($\mathcal{B}$).

  (1) Find orthogonal basis $\mathcal{Q} = \{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ for span($\mathcal{B}$) as follows:
  $\Big(\text{Here, } \mathcal{Q}_k := \{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k\} \text{ for } k = 1, \cdots, n-1\Big)$

$$
\begin{array}{lclcl}
\mathbf{q}_1 & := & \mathbf{v}_1 & = & \mathbf{v}_1 \\
\mathbf{q}_2 & := & \mathbf{v}_2 - \text{proj}_{span(\mathcal{Q}_1)}\mathbf{v}_2 & = & \mathbf{v}_2 - \text{proj}_{\mathbf{q}_1}\mathbf{v}_2 \\
\mathbf{q}_3 & := & \mathbf{v}_3 - \text{proj}_{span(\mathcal{Q}_2)}\mathbf{v}_3 & = & \mathbf{v}_3 - \text{proj}_{\mathbf{q}_1}\mathbf{v}_3 - \text{proj}_{\mathbf{q}_2}\mathbf{v}_3 \\
 & \vdots & \vdots & \vdots & \ddots \\
\mathbf{q}_n & := & \mathbf{v}_n - \text{proj}_{span(\mathcal{Q}_{n-1})}\mathbf{v}_n & = & \mathbf{v}_n - \text{proj}_{\mathbf{q}_1}\mathbf{v}_n - \text{proj}_{\mathbf{q}_2}\mathbf{v}_n - \cdots - \text{proj}_{\mathbf{q}_{n-1}}\mathbf{v}_n
\end{array}
$$

  (2) Find orthonormal basis $\widehat{\mathcal{Q}}$ from $\mathcal{Q}$ by normalizing each vector:

$$
\widehat{\mathbf{q}}_1 = \frac{\mathbf{q}_1}{||\mathbf{q}_1||}, \qquad \widehat{\mathbf{q}}_2 = \frac{\mathbf{q}_2}{||\mathbf{q}_2||}, \qquad \cdots, \qquad \widehat{\mathbf{q}}_n = \frac{\mathbf{q}_n}{||\mathbf{q}_n||}
$$

- **(CLASSICAL) GRAM-SCHMIDT ORTHONORMALIZATION, WITH EARLY NORMALIZATION:**

GIVEN: Induced-norm inner product space $(V, \langle\cdot,\cdot\rangle, ||\cdot||)$ & Basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$.

TASK: Find an orthonormal basis $\widehat{\mathcal{Q}} = \{\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2, \ldots, \widehat{\mathbf{q}}_n\}$ for span($\mathcal{B}$).

  (1) Find orthonormal basis $\widehat{\mathcal{Q}}$ as follows:
  $\big(\widehat{\mathcal{Q}}_k := \{\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2, \ldots, \widehat{\mathbf{q}}_k\} \text{ for } k = 1, \cdots, n-1\big)$

$$
\begin{array}{lclcll}
\mathbf{q}_1 & := & \mathbf{v}_1 & = & \mathbf{v}_1 & ;\ \widehat{\mathbf{q}}_1 := \mathbf{q}_1/||\mathbf{q}_1|| \\
\mathbf{q}_2 & := & \mathbf{v}_2 - \text{proj}_{span(\widehat{\mathcal{Q}}_1)}\mathbf{v}_2 & = & \mathbf{v}_2 - \text{proj}_{\widehat{\mathbf{q}}_1}\mathbf{v}_2 & ;\ \widehat{\mathbf{q}}_2 := \mathbf{q}_2/||\mathbf{q}_2|| \\
\mathbf{q}_3 & := & \mathbf{v}_3 - \text{proj}_{span(\widehat{\mathcal{Q}}_2)}\mathbf{v}_3 & = & \mathbf{v}_3 - \text{proj}_{\widehat{\mathbf{q}}_1}\mathbf{v}_3 - \text{proj}_{\widehat{\mathbf{q}}_2}\mathbf{v}_3 & ;\ \widehat{\mathbf{q}}_3 := \mathbf{q}_3/||\mathbf{q}_3|| \\
\vdots & & \vdots & & \ddots & \vdots \\
\mathbf{q}_n & := & \mathbf{v}_n - \text{proj}_{span(\widehat{\mathcal{Q}}_{n-1})}\mathbf{v}_n & = & \mathbf{v}_n - \text{proj}_{\widehat{\mathbf{q}}_1}\mathbf{v}_n - \cdots - \text{proj}_{\widehat{\mathbf{q}}_{n-1}}\mathbf{v}_n & ;\ \widehat{\mathbf{q}}_n := \mathbf{q}_n/||\mathbf{q}_n||
\end{array}
$$

REDUCED QR FACTORIZATION OF A MATRIX [LARSON 5.3]

- **MATRIX-VECTOR PRODUCTS ($A\mathbf{x}$) IN TERMS OF LINEAR COMBINATIONS OF COLUMNS:**

$$\text{Let } A := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \equiv \begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \mathbf{x} := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \underbrace{x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3}_{\substack{\textit{linear combination of} \\ \textit{the columns of } A}}$$

- **MATRIX-MATRIX PRODUCTS (AB) IN TERMS OF LINEAR COMBINATIONS OF COLUMNS:**

$$A := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \equiv \begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix}, \quad B := \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad C := \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} \equiv \begin{bmatrix} | & | \\ \mathbf{c}_1 & \mathbf{c}_2 \\ | & | \end{bmatrix}$$

$$C = AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \\ (a_{31}b_{11} + a_{32}b_{21}) & (a_{31}b_{12} + a_{32}b_{22}) \end{bmatrix} = \begin{bmatrix} | & | \\ (b_{11}\mathbf{a}_1 + b_{21}\mathbf{a}_2) & (b_{12}\mathbf{a}_1 + b_{22}\mathbf{a}_2) \\ | & | \end{bmatrix}$$

$$\therefore \; C = AB \implies \begin{cases} \mathbf{c}_1 = b_{11}\mathbf{a}_1 + b_{21}\mathbf{a}_2 \\ \mathbf{c}_2 = b_{12}\mathbf{a}_1 + b_{22}\mathbf{a}_2 \end{cases} \longleftarrow \text{Each column of } C \text{ is a linear combination of the columns of } A \text{ built from a column of } B$$

- **REDUCED QR FACTORIZATION VIA CLASSICAL GRAM-SCHMIDT w/ EARLY NORMING (CGS-EN):**

GIVEN: Tall or square ($m \geq n$) full column rank matrix $A_{m\times n} := \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{bmatrix}$.

TASK: Factor $A = \hat{Q}\hat{R}$ where $\hat{Q}_{m\times n}$ has orthonormal columns $\widehat{\mathbf{q}}_k$ and $\hat{R}_{n\times n} := [r_{ij}]_{n\times n}$ is upper triangular.

(1) Perform Classical Gram-Schmidt w/ early normalization on the columns of A:
$\big(\widehat{\mathcal{Q}}_k := \{\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2, \ldots, \widehat{\mathbf{q}}_k\}; \;\; k = 1, \cdots, n-1\big)$

$$\begin{array}{lllll}
\mathbf{q}_1 := \mathbf{a}_1 & = \mathbf{a}_1 & ; \; \widehat{\mathbf{q}}_1 := \mathbf{q}_1 / \underbrace{||\mathbf{q}_1||_2}_{r_{11}} \\
\mathbf{q}_2 := \mathbf{a}_2 - \text{proj}_{span(\widehat{\mathcal{Q}}_1)}\mathbf{a}_2 & = \mathbf{a}_2 - \underbrace{(\widehat{\mathbf{q}}_1^T\mathbf{a}_2)}_{r_{12}}\widehat{\mathbf{q}}_1 & ; \; \widehat{\mathbf{q}}_2 := \mathbf{q}_2 / \underbrace{||\mathbf{q}_2||_2}_{r_{22}} \\
\mathbf{q}_3 := \mathbf{a}_3 - \text{proj}_{span(\widehat{\mathcal{Q}}_2)}\mathbf{a}_3 & = \mathbf{a}_3 - \underbrace{(\widehat{\mathbf{q}}_1^T\mathbf{a}_3)}_{r_{13}}\widehat{\mathbf{q}}_1 - \underbrace{(\widehat{\mathbf{q}}_2^T\mathbf{a}_3)}_{r_{23}}\widehat{\mathbf{q}}_2 & ; \; \widehat{\mathbf{q}}_3 := \mathbf{q}_3 / \underbrace{||\mathbf{q}_3||_2}_{r_{33}} \\
\vdots & \vdots \quad \ddots & \vdots \\
\mathbf{q}_n := \mathbf{a}_n - \text{proj}_{span(\widehat{\mathcal{Q}}_{n-1})}\mathbf{a}_n & = \mathbf{a}_n - \underbrace{(\widehat{\mathbf{q}}_1^T\mathbf{a}_n)}_{r_{1n}}\widehat{\mathbf{q}}_1 - \cdots - \underbrace{(\widehat{\mathbf{q}}_{n-1}^T\mathbf{a}_n)}_{r_{n-1,n}}\widehat{\mathbf{q}}_{n-1} & ; \; \widehat{\mathbf{q}}_n := \mathbf{q}_n / \underbrace{||\mathbf{q}_n||_2}_{r_{nn}}
\end{array}$$

(2) Use the $\widehat{\mathbf{q}}_k$ vectors to build $\hat{Q}$ matrix and r_{ij} entries to build $\hat{R}$ matrix:

$$\hat{Q} = \begin{bmatrix} | & | & & | \\ \widehat{\mathbf{q}}_1 & \widehat{\mathbf{q}}_2 & \cdots & \widehat{\mathbf{q}}_n \\ | & | & & | \end{bmatrix}, \qquad \hat{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1,n\text{-}1} & r_{1n} \\ 0 & r_{22} & \cdots & r_{2,n\text{-}1} & r_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{n\text{-}1,n\text{-}1} & r_{n\text{-}1,n} \\ 0 & 0 & \cdots & 0 & r_{nn} \end{bmatrix}$$

- **REDUCED QR FACTORIZATION VIA CGS-EN (PROPERTIES):**

Let $A_{m\times n}$ be tall or square ($m \geq n$) and have full column rank. Let $A = \hat{Q}\hat{R}$ as Reduced QR via CGS-EN. Then:

(a) $\hat{Q}^T\hat{Q} = I_{n\times n}$ $\qquad$ (b) $r_{kk} \neq 0 \;\; \forall k$ $\qquad$ (c) $\hat{R}$ is invertible

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EX 5.3.1: Let vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and subspace $W = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \right\} \equiv \text{span}\{\mathbf{q}_1, \mathbf{q}_2\}$. Compute $\text{proj}_W \mathbf{v}$.

EX 5.3.2: Let orthonormal basis $\widehat{\mathcal{Q}} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4/5 \\ 3/5 \end{bmatrix}, \begin{bmatrix} 0 \\ -3/5 \\ 4/5 \end{bmatrix} \right\} \equiv \{\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2, \widehat{\mathbf{q}}_3\}$. Find $[\mathbf{x}]_{\widehat{\mathcal{Q}}}$ if $[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 20 \\ -5 \\ 50 \end{bmatrix}$.

EX 5.3.3: Let vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $\mathbf{u} = (4, -3, 0)^T$, $\mathbf{v} = (1, 2, 0)^T$, and $\mathbf{w} = (0, 0, 4)^T$.
Moreover, define inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$ as so: $\langle \mathbf{v}, \mathbf{w} \rangle := v_1 w_1 + v_2 w_2 + v_3 w_3 = \mathbf{v}^T \mathbf{w}$

Perform the classical Gram-Schmidt process to produce orthonormal basis $\widehat{\mathcal{Q}} := \{\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2, \widehat{\mathbf{q}}_3\}$ for $\operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

EX 5.3.4: Let functions $f, g, h \in C[0, \pi]$ such that $f(x) = 4$, $g(x) = x$, and $h(x) = \sin x$.

Moreover, define inner product $\langle \cdot, \cdot \rangle$ on $C[0, 1]$ as so: $\langle f, g \rangle := \int_0^{\pi} f(x)g(x)\ dx$

Perform the classical Gram-Schmidt process to determine an orthonormal basis $\widehat{\mathcal{Q}} := \{\widehat{q}_1, \widehat{q}_2, \widehat{q}_3\}$ for $\text{span}\{f, g, h\}$.