EX 5.4.1: Find the orthogonal complement of subspace $W = \text{span}\left\{\begin{bmatrix}1\\-2\\0\end{bmatrix}, \begin{bmatrix}-3\\5\\8\end{bmatrix}\right\}$.

Let $A = \begin{bmatrix} | & | \\ \mathbf{w}_1 & \mathbf{w}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 5 \\ 0 & 8 \end{bmatrix}$. Then, $W^\perp = \text{NulSp}(A^T)$

$$\left[\, A^T \,\middle|\, \vec{\mathbf{0}} \,\right] = \left[\begin{array}{ccc|c} \boxed{1} & -2 & 0 & 0 \\ -3 & 5 & 8 & 0 \end{array}\right] \xrightarrow{3R_1+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 0 & 0 \\ 0 & -1 & 8 & 0 \end{array}\right] \xrightarrow{(-1)R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 0 & 0 \\ 0 & \boxed{1} & -8 & 0 \end{array}\right]$$

$$\xrightarrow{2R_2+R_1\to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -16 & 0 \\ 0 & \boxed{1} & -8 & 0 \end{array}\right] = \left[\, \text{RREF}(A^T) \,\middle|\, \vec{\mathbf{0}} \,\right]$$

Interpret rows of $\left[\, \text{RREF}(A^T) \,\middle|\, \vec{\mathbf{0}} \,\right]$: Let $x_3 = t$. Then: $\begin{cases} x_1 - 16x_3 = 0 \\ x_2 - 8x_3 = 0 \end{cases} \implies \begin{cases} x_1 = 16t \\ x_2 = 8t \end{cases}$

$$\implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16t \\ 8t \\ t \end{bmatrix} = t\begin{bmatrix} 16 \\ 8 \\ 1 \end{bmatrix} \qquad \therefore \boxed{W^\perp = \text{span}\left\{\begin{bmatrix} 16 \\ 8 \\ 1 \end{bmatrix}\right\}}$$

EX 5.4.2: Let vector $\mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and subspace $W = \text{span}\left\{\begin{bmatrix}1\\-2\\0\end{bmatrix}, \begin{bmatrix}6\\3\\2\end{bmatrix}\right\} \equiv \text{span}\{\mathbf{q}_1, \mathbf{q}_2\}$. Compute $\text{proj}_W \mathbf{v}$.

$$\begin{aligned}
\text{proj}_W \mathbf{v} &= \text{proj}_{\mathbf{q}_1}\mathbf{v} + \text{proj}_{\mathbf{q}_2}\mathbf{v} \\
&= \frac{\mathbf{q}_1 \cdot \mathbf{v}}{\mathbf{q}_1 \cdot \mathbf{q}_1}\mathbf{q}_1 + \frac{\mathbf{q}_2^T \mathbf{v}}{\mathbf{q}_2^T \mathbf{q}_2}\mathbf{q}_2 \\
&= \frac{(1)(1) + (-2)(2) + (0)(3)}{(1)(1) + (-2)(-2) + (0)(0)}\mathbf{q}_1 + \frac{(6)(1) + (3)(2) + (2)(3)}{(6)(6) + (3)(3) + (2)(2)}\mathbf{q}_2 \\
&= \left(\frac{-3}{5}\right)\mathbf{q}_1 + \left(\frac{18}{49}\right)\mathbf{q}_2 \\
&= \left(-\frac{3}{5}\right)\begin{bmatrix}1\\-2\\0\end{bmatrix} + \left(\frac{18}{49}\right)\begin{bmatrix}6\\3\\2\end{bmatrix} \\
&= \begin{bmatrix}-3/5\\6/5\\0\end{bmatrix} + \begin{bmatrix}(6\cdot 18)/49\\(3\cdot 18)/49\\(2\cdot 18)/49\end{bmatrix} \\
&= \boxed{\begin{bmatrix}(-3\cdot 49 + 6\cdot 18\cdot 5)/(5\cdot 49)\\(6\cdot 49 + 3\cdot 18\cdot 5)/(5\cdot 49)\\(2\cdot 18)/49\end{bmatrix}} \leftarrow
\end{aligned}$$

If you want to simplify these fractions, have fun with that!
But leaving the fractions like this is acceptable for WeBWorK.

EX 5.4.3: Given the inconsistent linear system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(a) Find the least-squares solution $\mathbf{x}^*$ via the normal equations.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \qquad A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$\therefore$ The normal equations are: $A^TA\mathbf{x}^* = A^T\mathbf{b} \iff \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

Perform Gauss-Jordan Elimination on the augmented matrix:

$$\left[A^TA \mid A^T\mathbf{b}\right] = \left[\begin{array}{cc|c} 3 & -2 & 2 \\ -2 & 6 & -3 \end{array}\right] \xrightarrow[3R_2 \to R_2]{2R_1 \to R_1} \left[\begin{array}{cc|c} 6 & -4 & 4 \\ -6 & 18 & -9 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 6 & -4 & 4 \\ 0 & 14 & -5 \end{array}\right]$$

$$\xrightarrow[(\frac{1}{14})R_2 \to R_2]{(\frac{1}{6})R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & -2/3 & 2/3 \\ 0 & \boxed{1} & -5/14 \end{array}\right] \xrightarrow{(\frac{2}{3})R_2 + R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & 0 & 3/7 \\ 0 & \boxed{1} & -5/14 \end{array}\right] = \left[I \mid \mathbf{x}^*\right]$$

$\therefore$ $\boxed{\mathbf{x}^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 3/7 \\ -5/14 \end{bmatrix}}$ i.e. the "best-fit" line to the three points $(-2,1), (-1,1), (1,0)$ is $y = \frac{3}{7} - \frac{5}{14}x$.

(b) Find the minimum square-norm residual of the linear system: $\min_{\mathbf{x} \in \mathbb{R}^2} ||A\mathbf{x} - \mathbf{b}||_2^2$

$$A\mathbf{x}^* - \mathbf{b} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/7 \\ -5/14 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 16/14 \\ 11/14 \\ 1/14 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/14 \\ -3/14 \\ 1/14 \end{bmatrix}$$

$$\min_{\mathbf{x} \in \mathbb{R}^2} ||A\mathbf{x} - \mathbf{b}||_2^2 = ||A\mathbf{x}^* - \mathbf{b}||_2^2 = \left|\left|\begin{bmatrix} 2/14 \\ -3/14 \\ 1/14 \end{bmatrix}\right|\right|_2^2 = \begin{bmatrix} 2/14 \\ -3/14 \\ 1/14 \end{bmatrix} \cdot \begin{bmatrix} 2/14 \\ -3/14 \\ 1/14 \end{bmatrix} = \left(\frac{2}{14}\right)^2 + \left(-\frac{3}{14}\right)^2 + \left(\frac{1}{14}\right)^2 = \boxed{\frac{1}{14}}$$

This essentially means that the sum of squares of the vertical errors of the "best-fit" line to the three points is $\frac{1}{14}$.

(c) Find the projection matrix $\bar{P}$ onto the column space of A.

Since A^TA is a 2x2 matrix, use determinant-adjoint formula (Ch2/Ch3) to find its inverse.

In general, find inverses of a 3x3 or larger A^TA matrix via Gauss-Jordan Elimation of augmented matrix: $\left[A^TA \mid I\right]$

$$\therefore \bar{P} = A(A^TA)^{-1}A^T = \frac{1}{14} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 1 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 2 & -4 \\ 4 & -1 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 1 \end{bmatrix} = \boxed{\frac{1}{14} \begin{bmatrix} 10 & 6 & -2 \\ 6 & 5 & 3 \\ -2 & 3 & 13 \end{bmatrix}}$$

(d) Find the best approximation $\mathbf{b}^* \in \text{ColSp}(A)$ to the RHS vector $\mathbf{b}$.

(Option #1) Use least-squares solution: $\mathbf{b}^* = A\mathbf{x}^* = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/7 \\ -5/14 \end{bmatrix} = \boxed{\begin{bmatrix} 16/14 \\ 11/14 \\ 1/14 \end{bmatrix}}$

(Option #2) Use projection matrix: $\mathbf{b}^* = \bar{P}\mathbf{b} = \frac{1}{14} \begin{bmatrix} 10 & 6 & -2 \\ 6 & 5 & 3 \\ -2 & 3 & 13 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 16 \\ 11 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 16/14 \\ 11/14 \\ 1/14 \end{bmatrix}}$

This essentially means that $A\mathbf{x}^* = \mathbf{b}^*$ is the "closest" consistent linear system to the inconsistent system $A\mathbf{x} = \mathbf{b}$.

EX 5.4.4: Given the inconsistent linear system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(a) Perform the Reduced QR Factorization of A using Classical Gram-Schmidt with early normalization.

$$\mathbf{q}_1 := \mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \implies r_{11} := ||\mathbf{q}_1||_2 = \sqrt{1^2+1^2+1^2} = \sqrt{3} \implies \hat{\mathbf{q}}_1 := \frac{\mathbf{q}_1}{r_{11}} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\mathbf{q}_2 := \mathbf{a}_2 - \text{proj}_{span(\hat{Q}_1)}\mathbf{a}_2 = \mathbf{a}_2 - \text{proj}_{\hat{\mathbf{q}}_1}\mathbf{a}_2 = \mathbf{a}_2 - \underbrace{(\hat{\mathbf{q}}_1^T\mathbf{a}_2)}_{r_{12}}\hat{\mathbf{q}}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} - \underbrace{\left(\frac{-2}{\sqrt{3}}\right)}_{r_{12}}\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} -4/3 \\ -1/3 \\ 5/3 \end{bmatrix}$$

$$\implies r_{22} := ||\mathbf{q}_2||_2 = \sqrt{\left(\frac{-4}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{5}{3}\right)^2} = \frac{\sqrt{42}}{3} \implies \hat{\mathbf{q}}_2 := \frac{\mathbf{q}_2}{r_{22}} = \frac{3}{\sqrt{42}}\begin{bmatrix} -4/3 \\ -1/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -4/\sqrt{42} \\ -1/\sqrt{42} \\ 5/\sqrt{42} \end{bmatrix}$$

$$\therefore \hat{Q} := \begin{bmatrix} | & | \\ \hat{\mathbf{q}}_1 & \hat{\mathbf{q}}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{42} \\ 1/\sqrt{3} & -1/\sqrt{42} \\ 1/\sqrt{3} & 5/\sqrt{42} \end{bmatrix} \quad \text{and} \quad \hat{R} = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -2/\sqrt{3} \\ 0 & \sqrt{42}/3 \end{bmatrix}$$

$$\therefore \boxed{A = \hat{Q}\hat{R} = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{42} \\ 1/\sqrt{3} & -1/\sqrt{42} \\ 1/\sqrt{3} & 5/\sqrt{42} \end{bmatrix}\begin{bmatrix} \sqrt{3} & -2/\sqrt{3} \\ 0 & \sqrt{42}/3 \end{bmatrix}} \longleftarrow \text{SANITY CHECK: } \hat{Q}^T\hat{Q} = I_{2\times 2}$$

(b) Find the projection matrix $\bar{P}$ onto the column space of A.

$$\bar{P} := \hat{Q}\hat{Q}^T = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{42} \\ 1/\sqrt{3} & -1/\sqrt{42} \\ 1/\sqrt{3} & 5/\sqrt{42} \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{bmatrix} = \begin{bmatrix} 5/7 & 3/7 & -1/7 \\ 3/7 & 5/14 & 3/14 \\ -1/7 & 3/14 & 13/14 \end{bmatrix} = \boxed{\frac{1}{14}\begin{bmatrix} 10 & 6 & -2 \\ 6 & 5 & 3 \\ -2 & 3 & 13 \end{bmatrix}}$$

(c) Find the best approximation $\mathbf{b}^* \in \text{ColSp}(A)$ to the RHS vector $\mathbf{b}$.

$$\mathbf{b}^* = \bar{P}\mathbf{b} = \frac{1}{14}\begin{bmatrix} 10 & 6 & -2 \\ 6 & 5 & 3 \\ -2 & 3 & 13 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 16 \\ 11 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 16/14 \\ 11/14 \\ 1/14 \end{bmatrix}}$$

(d) Find the minimum square-norm residual of the linear system: $\min_{\mathbf{x}\in\mathbb{R}^2} ||A\mathbf{x} - \mathbf{b}||_2^2$

$$\min_{\mathbf{x}\in\mathbb{R}^2} ||A\mathbf{x} - \mathbf{b}||_2^2 = ||\mathbf{b} - \bar{P}\mathbf{b}||_2^2 = \left|\left|\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 16/14 \\ 11/14 \\ 1/14 \end{bmatrix}\right|\right|_2^2 = \left|\left|\begin{bmatrix} -2/14 \\ 3/14 \\ -1/14 \end{bmatrix}\right|\right|_2^2 = \begin{bmatrix} -2/14 \\ 3/14 \\ -1/14 \end{bmatrix} \cdot \begin{bmatrix} -2/14 \\ 3/14 \\ -1/14 \end{bmatrix} = \boxed{\frac{1}{14}}$$

(e) Find the least-squares solution $\mathbf{x}^*$ via the Reduced QR Factorization performed in part (a).

Use the Reduced QR Factorization to form upper-triangular linear system:

$$A\mathbf{x} = \mathbf{b} \overset{QR}{\iff} \hat{Q}\hat{R}\mathbf{x}^* = \mathbf{b} \overset{\hat{Q}^T}{\iff} \hat{Q}^T\hat{Q}\hat{R}\mathbf{x}^* = \hat{Q}^T\mathbf{b} \overset{I_{2\times 2}}{\iff} \hat{R}\mathbf{x}^* = \hat{Q}^T\mathbf{b}$$

$$\hat{Q}^T\mathbf{b} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ -5\sqrt{42} \end{bmatrix}$$

$$\hat{R}\mathbf{x}^* = \hat{Q}^T\mathbf{b} \iff \begin{bmatrix} \sqrt{3} & -2/\sqrt{3} \\ 0 & \sqrt{42}/3 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ -5\sqrt{42} \end{bmatrix}$$

Back-solve upper-triangular linear system:

$$\frac{\sqrt{42}}{3}x_2^* = -\frac{5}{\sqrt{42}} \implies x_2^* = -\frac{15}{42} = -\frac{5}{14}; \ (\sqrt{3})x_1^* - \frac{2}{\sqrt{3}}x_2^* = \frac{2}{\sqrt{3}} \implies (\sqrt{3})x_1^* - \left(\frac{2}{\sqrt{3}}\right)\left(-\frac{5}{14}\right) = \frac{2}{\sqrt{3}} \implies x_1^* = \frac{3}{7}$$

$$\therefore \mathbf{x}^* := \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \boxed{\begin{bmatrix} 3/7 \\ -5/14 \end{bmatrix}}$$

EX 5.4.5: Given the inconsistent linear system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(a) Perform the Full QR Factorization of A using Classical Gram-Schmidt with early normalization.

From EX 5.4.4: $\hat{Q} := \begin{bmatrix} | & | \\ \hat{\mathbf{q}}_1 & \hat{\mathbf{q}}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{42} \\ 1/\sqrt{3} & -1/\sqrt{42} \\ 1/\sqrt{3} & 5/\sqrt{42} \end{bmatrix}$ and $\hat{R} = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -2/\sqrt{3} \\ 0 & \sqrt{42}/3 \end{bmatrix}$

Produce a basis $\{\mathbf{a}_{n+1}, \mathbf{a}_{n+2}, \cdots, \mathbf{a}_m\}$ for $\text{ColSp}(A)^\perp \overset{FTLA}{=} \text{NulSp}(A^T)$:

$$\left[A^T \,\middle|\, \vec{\mathbf{0}} \right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array} \right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \end{array} \right] = \left[\text{RREF}(A^T) \,\middle|\, \vec{\mathbf{0}} \right]$$

There is one column of $\text{RREF}(A^T)$ without a pivot, meaning there's one free variable.

Let $v_3 = t$. Then, $v_2 + 3t = 0 \implies v_2 = -3t$ and $v_1 - 2t = 0 \implies v_1 = 2t \implies \mathbf{v} := \begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

$$\therefore\ \text{ColSp}(A)^\perp = \text{Span}\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right\} \implies \mathbf{a}_3 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \implies \hat{\mathbf{q}}_3 := \frac{\mathbf{a}_3}{||\mathbf{a}_3||_2} = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{14} \\ -3/\sqrt{14} \\ 1/\sqrt{14} \end{bmatrix}$$

Form Q by augmenting $\hat{Q}_r$ to $\hat{Q}$, and form R by augmenting compatible zero matrix below $\hat{R}$:

$$\therefore\ \hat{Q}_r := \begin{bmatrix} | \\ \hat{\mathbf{q}}_3 \\ | \end{bmatrix} = \begin{bmatrix} 2/\sqrt{14} \\ -3/\sqrt{14} \\ 1/\sqrt{14} \end{bmatrix}, \quad Q := \begin{bmatrix} \hat{Q} & \hat{Q}_r \end{bmatrix} = \begin{bmatrix} | & | & | \\ \hat{\mathbf{q}}_1 & \hat{\mathbf{q}}_2 & \hat{\mathbf{q}}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{42} & 2/\sqrt{14} \\ 1/\sqrt{3} & -1/\sqrt{42} & -3/\sqrt{14} \\ 1/\sqrt{3} & 5/\sqrt{42} & 1/\sqrt{14} \end{bmatrix}$$

$$\therefore\ R := \begin{bmatrix} \hat{R}_{2\times 2} \\ O_{1\times 2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -2/\sqrt{3} \\ 0 & \sqrt{42}/3 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{A = QR = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{42} & 2/\sqrt{14} \\ 1/\sqrt{3} & -1/\sqrt{42} & -3/\sqrt{14} \\ 1/\sqrt{3} & 5/\sqrt{42} & 1/\sqrt{14} \end{bmatrix} \begin{bmatrix} \sqrt{3} & -2/\sqrt{3} \\ 0 & \sqrt{42}/3 \\ 0 & 0 \end{bmatrix}}$$

(b) Find the projection matrix $\bar{P}$ onto $\text{ColSp}(A)$.

$$\bar{P} := \hat{Q}\hat{Q}^T = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{42} \\ 1/\sqrt{3} & -1/\sqrt{42} \\ 1/\sqrt{3} & 5/\sqrt{42} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{bmatrix} = \begin{bmatrix} 5/7 & 3/7 & -1/7 \\ 3/7 & 5/14 & 3/14 \\ -1/7 & 3/14 & 13/14 \end{bmatrix} = \boxed{\frac{1}{14} \begin{bmatrix} 10 & 6 & -2 \\ 6 & 5 & 3 \\ -2 & 3 & 13 \end{bmatrix}}$$

(c) Find the best approximation $\mathbf{b}^* \in \text{ColSp}(A)$ to the RHS vector $\mathbf{b}$.

$$\mathbf{b}^* = \bar{P}\mathbf{b} = \frac{1}{14} \begin{bmatrix} 10 & 6 & -2 \\ 6 & 5 & 3 \\ -2 & 3 & 13 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 16 \\ 11 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 16/14 \\ 11/14 \\ 1/14 \end{bmatrix}}$$

(d) Find the projection matrix $\bar{P}_r$ onto $\text{ColSp}(A)^\perp$.

$$\bar{P}_r := \hat{Q}_r\hat{Q}_r^T = \begin{bmatrix} 2/\sqrt{14} \\ -3/\sqrt{14} \\ 1/\sqrt{14} \end{bmatrix} \begin{bmatrix} 2/\sqrt{14} & -3/\sqrt{14} & 1/\sqrt{14} \end{bmatrix} = \begin{bmatrix} 4/14 & -6/14 & 2/14 \\ -6/14 & 9/14 & -3/14 \\ 2/14 & -3/14 & 1/14 \end{bmatrix} = \boxed{\frac{1}{14} \begin{bmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{bmatrix}}$$

(e) Find the minimum square-norm residual of the linear system: $\min_{\mathbf{x}\in\mathbb{R}^2} ||A\mathbf{x} - \mathbf{b}||_2^2$

$$\bar{P}_r\mathbf{b} = \frac{1}{14} \begin{bmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2/14 \\ 3/14 \\ -1/14 \end{bmatrix}$$

$$\min_{\mathbf{x}\in\mathbb{R}^2} ||A\mathbf{x} - \mathbf{b}||_2^2 = ||\bar{P}_r\mathbf{b}||_2^2 = \left\| \begin{bmatrix} -2/14 \\ 3/14 \\ -1/14 \end{bmatrix} \right\|_2^2 = \begin{bmatrix} -2/14 \\ 3/14 \\ -1/14 \end{bmatrix} \cdot \begin{bmatrix} -2/14 \\ 3/14 \\ -1/14 \end{bmatrix} = \boxed{\frac{1}{14}}$$

(f) Find the least-squares solution $\mathbf{x}^*$ via the Full QR Factorization performed in part (a).

Use the Reduced QR Factorization from EX 5.4.4: $\mathbf{x}^* := \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \boxed{\begin{bmatrix} 3/7 \\ -5/14 \end{bmatrix}}$

EX 5.4.6: Perform the Full QR Factorization using Classical Gram-Schmidt with early normalization of $A = \begin{bmatrix} 3 & 0 & 2 \\ -3 & 2 & -1 \\ 3 & -1 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

1^{st}, Perform Classical Gram-Schmidt w/ early normalization on the columns of A: $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \xrightarrow{CGS\text{-}EN} \{\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2, \widehat{\mathbf{q}}_3\}$

$$\mathbf{q}_1 := \mathbf{a}_1 = (3,\ -3,\ 3,\ 0,\ 0)^T$$

$$\widehat{\mathbf{q}}_1 = \frac{\mathbf{q}_1}{r_{11}} = \frac{\mathbf{q}_1}{||\mathbf{q}_1||_2} = \frac{\mathbf{q}_1}{3\sqrt{3}} = (1/\sqrt{3},\ -1/\sqrt{3},\ 1/\sqrt{3},\ 0,\ 0)^T$$

$$\mathbf{q}_2 := \mathbf{a}_2 - \text{proj}_{\widehat{\mathbf{q}}_1}\mathbf{a}_2 = \mathbf{a}_2 - \underbrace{(\widehat{\mathbf{q}}_1^T\mathbf{a}_2)}_{r_{12}}\widehat{\mathbf{q}}_1 = \mathbf{a}_2 - \underbrace{(-\sqrt{3})}_{r_{12}}\widehat{\mathbf{q}}_1 = (1,\ 1,\ 0,\ -1,\ 0)^T$$

$$\widehat{\mathbf{q}}_2 = \frac{\mathbf{q}_2}{r_{22}} = \frac{\mathbf{q}_2}{||\mathbf{q}_2||_2} = \frac{\mathbf{q}_2}{\sqrt{3}} = (1/\sqrt{3},\ 1/\sqrt{3},\ 0,\ -1/\sqrt{3},\ 0)^T$$

$$\mathbf{q}_3 := \mathbf{a}_3 - \text{proj}_{\widehat{\mathbf{q}}_1}\mathbf{a}_3 - \text{proj}_{\widehat{\mathbf{q}}_2}\mathbf{a}_3 = \mathbf{a}_3 - \underbrace{(\widehat{\mathbf{q}}_1^T\mathbf{a}_3)}_{r_{13}}\widehat{\mathbf{q}}_1 - \underbrace{(\widehat{\mathbf{q}}_2^T\mathbf{a}_3)}_{r_{23}}\widehat{\mathbf{q}}_2 = \mathbf{a}_3 - \underbrace{(2\sqrt{3})}_{r_{13}}\widehat{\mathbf{q}}_1 - \underbrace{(-\sqrt{3})}_{r_{23}}\widehat{\mathbf{q}}_2 = (1,\ 2,\ 1,\ 3,\ 1)^T$$

$$\widehat{\mathbf{q}}_3 = \frac{\mathbf{q}_3}{r_{33}} = \frac{\mathbf{q}_3}{||\mathbf{q}_3||_2} = \frac{\mathbf{q}_3}{4} = (1/4,\ 2/4,\ 1/4,\ 3/4,\ 1/4)^T$$

$$A = \hat{Q}\hat{R} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/4 \\ -1/\sqrt{3} & 1/\sqrt{3} & 2/4 \\ 1/\sqrt{3} & 0 & 1/4 \\ 0 & -1/\sqrt{3} & 3/4 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 3\sqrt{3} & -\sqrt{3} & 2\sqrt{3} \\ 0 & \sqrt{3} & -\sqrt{3} \\ 0 & 0 & 4 \end{bmatrix} \longleftarrow \text{SANITY CHECK: } \hat{Q}^T\hat{Q} = I_{3\times 3}$$

2^{nd}, Produce a basis $\{\mathbf{a}_4, \mathbf{a}_5\}$ for $\text{ColSp}(A)^\perp \overset{FTLA}{=} \text{NulSp}(A^T)$:

$$\left[\, A^T \,\middle|\, \vec{\mathbf{0}} \,\right] = \left[\begin{array}{ccccc|c} 3 & -3 & 3 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ 2 & -1 & 3 & 4 & 1 & 0 \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & -2 & -1/3 & 0 \\ 0 & \boxed{1} & 0 & 1 & 1/3 & 0 \\ 0 & 0 & \boxed{1} & 3 & 2/3 & 0 \end{array}\right] = \left[\, \text{RREF}(A^T) \,\middle|\, \vec{\mathbf{0}} \,\right]$$

There are two columns of $\text{RREF}(A^T)$ without a pivot, meaning there are two free variables: x_4, x_5

Assign a **unique parameter** to each **free variable** and solve for each component: x_1, x_2, x_3, x_4, x_5

$$\begin{cases} x_1 - 2x_4 - \frac{1}{3}x_5 &= 0 \\ x_2 + x_4 + \frac{1}{3}x_5 &= 0 \\ x_3 + 3x_4 + \frac{2}{3}x_5 &= 0 \\ x_4 &:= s \\ x_5 &:= 3t \end{cases} \implies \begin{cases} x_1 &= 2s + \frac{1}{3}(3t) \\ x_2 &= -s - \frac{1}{3}(3t) \\ x_3 &= -3s - \frac{2}{3}(3t) \\ x_4 &= s \\ x_5 &= 3t \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\underbrace{\begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{a}_4} + t\underbrace{\begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \\ 3 \end{bmatrix}}_{\mathbf{a}_5} \in \text{NulSp}(A^T)$$

3^{rd}, Perform Classical Gram-Schmidt w/ early normalization on the basis of $\text{ColSp}(A)^\perp$: $\{\mathbf{a}_4, \mathbf{a}_5\} \xrightarrow{CGS\text{-}EN} \{\widehat{\mathbf{q}}_4, \widehat{\mathbf{q}}_5\}$

$$\mathbf{q}_4 := \mathbf{a}_4 = (2,\ -1,\ -3,\ 1,\ 0)^T$$

$$\widehat{\mathbf{q}}_4 = \frac{\mathbf{q}_4}{||\mathbf{q}_4||_2} = \frac{\mathbf{q}_4}{\sqrt{15}} = (2/\sqrt{15},\ -1/\sqrt{15},\ -3/\sqrt{15},\ 1/\sqrt{15},\ 0)^T$$

$$\mathbf{q}_5 := \mathbf{a}_5 - \text{proj}_{\widehat{\mathbf{q}}_4}\mathbf{a}_5 = \mathbf{a}_5 - (\widehat{\mathbf{q}}_4^T\mathbf{a}_5)\widehat{\mathbf{q}}_4 = \mathbf{a}_5 - \left(\frac{9}{\sqrt{15}}\right)\widehat{\mathbf{q}}_4 = (-1/5,\ -2/5,\ -1/5,\ -3/5,\ 15/5)^T$$

$$\widehat{\mathbf{q}}_5 = \frac{\mathbf{q}_5}{||\mathbf{q}_5||_2} = \frac{\mathbf{q}_5}{\sqrt{240}/5} = (-1/\sqrt{240},\ -2/\sqrt{240},\ -1/\sqrt{240},\ -3/\sqrt{240},\ 15/\sqrt{240})^T$$

$$\therefore\ \hat{Q}_r := \begin{bmatrix} | & | \\ \widehat{\mathbf{q}}_4 & \widehat{\mathbf{q}}_5 \\ | & | \end{bmatrix} = \begin{bmatrix} 2/\sqrt{15} & -1/\sqrt{240} \\ -1/\sqrt{15} & -2/\sqrt{240} \\ -3/\sqrt{15} & -1/\sqrt{240} \\ 1/\sqrt{15} & -3/\sqrt{240} \\ 0 & 15/\sqrt{240} \end{bmatrix}$$

4^{th}, Form Q by augmenting $\hat{Q}_r$ to $\hat{Q}$, and form R by augmenting compatible zero matrix below $\hat{R}$:

$$\therefore\ A = QR = \begin{bmatrix} \hat{Q}_{5\times 3} & \hat{Q}_{r,5\times 2} \end{bmatrix} \begin{bmatrix} \hat{R}_{3\times 3} \\ O_{2\times 3} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/4 & 2/\sqrt{15} & -1/\sqrt{240} \\ -1/\sqrt{3} & 1/\sqrt{3} & 2/4 & -1/\sqrt{15} & -2/\sqrt{240} \\ 1/\sqrt{3} & 0 & 1/4 & -3/\sqrt{15} & -1/\sqrt{240} \\ 0 & -1/\sqrt{3} & 3/4 & 1/\sqrt{15} & -3/\sqrt{240} \\ 0 & 0 & 1/4 & 0 & 15/\sqrt{240} \end{bmatrix} \begin{bmatrix} 3\sqrt{3} & -\sqrt{3} & 2\sqrt{3} \\ 0 & \sqrt{3} & -\sqrt{3} \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$