SUBSPACE PROJECTIONS, DIRECT SUMS, ORTHOGONAL COMPLEMENTS [LARSON 5.4]

- **(ORTHOGONAL) PROJECTION ONTO A SUBSPACE (DEFINITION):**

  Let $\mathcal{Q} = \{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ be an orthogonal basis for subspace W of $\mathbb{R}^n$.

  Then the **(orthogonal) projection of vector $\mathbf{v} \in \mathbb{R}^n$ onto subspace W** is:

$$\text{proj}_W \mathbf{v} := \text{proj}_{span(\mathcal{Q})} \mathbf{v} = \text{proj}_{\mathbf{q}_1} \mathbf{v} + \text{proj}_{\mathbf{q}_2} \mathbf{v} + \cdots + \text{proj}_{\mathbf{q}_n} \mathbf{v}$$

- **ORTHOGONAL SUBSPACES OF $\mathbb{R}^n$ (DEFINITION):**

  Subspaces S_1, S_2 of $\mathbb{R}^n$ are **orthogonal**, denoted $S_1 \perp S_2$, if $\quad \mathbf{v}_1^T \mathbf{v}_2 = 0 \quad \forall \mathbf{v}_1 \in S_1, \forall \mathbf{v}_2 \in S_2$

  i.e. vectors in one subspace are orthogonal to vectors in the other subspace.

- **ORTHOGONAL COMPLEMENTS (DEFINITION):**

  Let W be a subspace of Euclidean inner product space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_2)$.

  Then the **orthogonal complement of** W, denoted $W^\perp$, is $\quad W^\perp := \{\mathbf{w}^\perp \in \mathbb{R}^n : \langle \mathbf{w}, \mathbf{w}^\perp \rangle_2 = 0 \ \ \forall \mathbf{w} \in W\}$

  i.e. $W^\perp$ is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to all vectors in W. Clearly, by definition, $W \perp W^\perp$.

  SPECIAL CASES: $(\mathbb{R}^n)^\perp = \{\vec{\mathbf{0}}\}$ AND $\{\vec{\mathbf{0}}\}^\perp = \mathbb{R}^n$

- **FINDING THE ORTHOGONAL COMPLEMENT (PROCEDURE):**

  GIVEN: Subspace W of $\mathbb{R}^n$ such that $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p\}$.

  TASK: Find orthogonal complement $W^\perp$.

  (1) Form matrix $A = \begin{bmatrix} | & | & & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_p \\ | & | & & | \end{bmatrix}$

  (2) $W^\perp = \text{NulSp}(A^T) \implies [\ A^T \mid \vec{\mathbf{0}}\ ] \xrightarrow{Gauss-Jordan} [\ \text{RREF}(A^T) \mid \vec{\mathbf{0}}\ ]$

- **DIRECT SUMS OF SUBSPACES OF $\mathbb{R}^n$ (DEFINITION):** Let W_1, W_2 be two subspaces of $\mathbb{R}^n$.

  Then $\mathbb{R}^n$ is the **direct sum** of W_1 & W_2, written $\mathbb{R}^n = W_1 \oplus W_2$, if

  $\mathbf{v} \in \mathbb{R}^n$ can be uniquely written as $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in W_1$ & $\mathbf{w}_2 \in W_2$.

  i.e. each vector in $\mathbb{R}^n$ can be uniquely written as a sum of a vector from W_1 and a vector from W_2.

- **PROPERTIES OF ORTHOGONAL COMPLEMENTS:**

  Let W be a subspace of Euclidean induced-norm inner product space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_2, || \cdot ||_2)$. Then:

  (i) $W^\perp$ is also a subspace of $\mathbb{R}^n$
  (ii) $W \cap W^\perp = \{\vec{\mathbf{0}}\}$
  (iii) $\mathbb{R}^n = W \oplus W^\perp$
  (iv) $\dim(\mathbb{R}^n) = \dim(W) + \dim(W^\perp)$
  (v) $(W^\perp)^\perp = W$

SUBSPACES OF A MATRIX, FULL-RANK LEAST-SQUARES [LARSON 5.4]

- **FUNDAMENTAL SUBSPACES OF A MATRIX:** Let matrix $A \in \mathbb{R}^{m\times n}$ such that $\text{rank}(A) = r$.

  Then the fundamental subspaces of A are related as so:

(i)	$\text{RowSp}(A) = \text{ColSp}(A^T)$		(iv)	$\dim \text{ColSp}(A) = \dim \text{ColSp}(A^T) = r$
(ii)	$\text{ColSp}(A)^\perp = \text{NulSp}(A^T)$		(v)	$\mathbb{R}^m = \text{ColSp}(A) \oplus \text{NulSp}(A^T)$
(iii)	$\text{ColSp}(A^T)^\perp = \text{NulSp}(A)$		(vi)	$\mathbb{R}^n = \text{ColSp}(A^T) \oplus \text{NulSp}(A)$

- **PYTHAGOREAN THEOREM FOR ORTHOGONAL VECTORS (PTFOV):**

$$\text{Vectors } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \text{ are orthogonal} \iff ||\mathbf{u}+\mathbf{v}||_2^2 = ||\mathbf{u}||_2^2 + ||\mathbf{v}||_2^2$$

- **BEST APPROXIMATION THEOREM:**

  Let W be a subspace of $\mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ s.t. $\mathbf{v} \notin W$. Then:

$$||\mathbf{v} - \text{proj}_W \mathbf{v}||_2 < ||\mathbf{v} - \mathbf{u}||_2 \qquad \forall \mathbf{u} \in S \text{ s.t. } \mathbf{u} \neq \text{proj}_W \mathbf{v}$$

  i.e. the projection of $\mathbf{v}$ onto W is the "closest" vector in W to $\mathbf{v}$ which is not in W.

  $\text{proj}_W \mathbf{v}$ is called the **best approximation** to $\mathbf{v}$ in subspace W.

- **THE LEAST-SQUARES PROBLEM & SOLUTION (DEFINITION):**

  Let $A \in \mathbb{R}^{m\times n}$ such that $m > n$ and $\mathbf{b} \notin \text{ColSp}(A)$ such that linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent & overdetermined.

  Then the **least-squares problem** is to find $\mathbf{x} \in \mathbb{R}^n$ s.t. $||\mathbf{b} - A\mathbf{x}||_2^2$ is minimized: $\min\limits_{\mathbf{x}\in\mathbb{R}^n} ||\mathbf{b} - A\mathbf{x}||_2^2$

  REMARK: Vector $(\mathbf{b} - A\mathbf{x})$ is called the **residual** of the linear system.

  Vector $\mathbf{x}^* \in \mathbb{R}^n$ is a **least-squares solution** to $A\mathbf{x} = \mathbf{b}$ if: $\min\limits_{\mathbf{x}\in\mathbb{R}^n} ||\mathbf{b} - A\mathbf{x}||_2^2 = ||\mathbf{b} - A\mathbf{x}^*||_2^2$

  i.e. $||\mathbf{b} - A\mathbf{x}^*||_2^2$ is the minimum square-norm of the residual.

- **FULL-RANK LEAST-SQUARES PROCEDURE USING NORMAL EQUATIONS:**

  GIVEN: $m \times n$ $(m \geq n)$ linear system $A\mathbf{x} = \mathbf{b}$, full column rank A, $\mathbf{b} \notin \text{ColSp}(A)$.

  TASK: Find Least-Squares Solution $\mathbf{x}^*$ s.t. $||\mathbf{b} - A\mathbf{x}||_2^2$ is minimized.

  (1) Form **normal equations** for $\mathbf{x}^*$: $A^T A\mathbf{x}^* = A^T\mathbf{b}$

  (2) Solve normal equations for $\mathbf{x}^*$: $\left[\, A^T A \mid A^T\mathbf{b} \,\right] \xrightarrow{Gauss-Jordan} [\, I \mid \mathbf{x}^* \,]$

  (3) Minimize square-norm of Residual: $\min\limits_{\mathbf{x}\in\mathbb{R}^n} ||\mathbf{b} - A\mathbf{x}||_2^2 = ||\mathbf{b} - A\mathbf{x}^*||_2^2$

  (4) Find Projection Matrix onto ColSp(A): $\bar{P} = A(A^T A)^{-1}A^T$

  (5) Find Best Approximation $\mathbf{b}^* \in \text{ColSp}(A)$ to $\mathbf{b}$: $\mathbf{b}^* = \bar{P}\mathbf{b} = A\mathbf{x}^*$

- **FULL-RANK LEAST-SQUARES PROCEDURE USING REDUCED QR FACTORIZATION:**

  GIVEN: $m \times n$ $(m \geq n)$ linear system $A\mathbf{x} = \mathbf{b}$, full column rank A, $\mathbf{b} \notin \text{ColSp}(A)$.

  TASK: Find Least-Squares Solution $\mathbf{x}^*$ s.t. $||\mathbf{b} - A\mathbf{x}||_2^2$ is minimized.

  (1) Perform Reduced QR Factorization using CGS-EN: $A = \hat{Q}\hat{R}$

  (Recall that with Reduced QR, $\hat{Q}$ is $m \times n$ and $\hat{R}$ is $n \times n$.)

  (2) Find Projection Matrix onto ColSp(A): $\bar{P} = \hat{Q}\hat{Q}^T$

  (3) Find Best Approximation $\mathbf{b}^* \in \text{ColSp}(A)$ to $\mathbf{b}$: $\mathbf{b}^* = \bar{P}\mathbf{b}$

  (4) Minimize square-norm of Residual: $\min\limits_{\mathbf{x}\in\mathbb{R}^n} ||\mathbf{b} - A\mathbf{x}||_2^2 = ||\mathbf{b} - \bar{P}\mathbf{b}||_2^2$

  (5) Back-solve linear system $\hat{R}\mathbf{x}^* = \hat{Q}^T\mathbf{b}$ for $\mathbf{x}^*$.

FULL QR, FULL-RANK LEAST-SQUARES, ORTHOGONAL MATRICES

- **FULL QR FACTORIZATION VIA CLASSICAL GRAM-SCHMIDT w/ EARLY NORMING (CGS-EN):**

GIVEN: Tall or square ($m \geq n$) full column rank matrix $A_{m\times n}$ with columns $\mathbf{a}_k$.

TASK: Factor $A = QR$ where $Q_{m\times m}$ has orthonormal columns $\widehat{\mathbf{q}}_k$ and $R_{m\times n}$ is upper triangular.

(1) Perform Classical Gram-Schmidt w/ early normalization on the columns of A, $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$:

$$\hat{Q} = \begin{bmatrix} | & | & & | \\ \widehat{\mathbf{q}}_1 & \widehat{\mathbf{q}}_2 & \cdots & \widehat{\mathbf{q}}_n \\ | & | & & | \end{bmatrix}, \qquad \hat{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ 0 & r_{22} & r_{23} & \cdots & r_{2n} \\ 0 & 0 & r_{33} & \cdots & r_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{nn} \end{bmatrix}$$

(2) Produce a basis $\{\mathbf{a}_{n+1}, \mathbf{a}_{n+2}, \cdots, \mathbf{a}_m\}$ for orthogonal complement of column space of A:

$$\left[A^T \,\middle|\, \vec{\mathbf{0}}\right] \xrightarrow{Gauss-Jordan} \left[\text{RREF}(A^T) \,\middle|\, \vec{\mathbf{0}}\right]$$

(3) Perform CGS-EN on the basis $\{\mathbf{a}_{n+1}, \mathbf{a}_{n+2}, \cdots, \mathbf{a}_m\}$, resulting in $\hat{Q}_r$ matrix.

(4) Form Q by augmenting $\hat{Q}_r$ to $\hat{Q}$, and form R by augmenting zero matrix below $\hat{R}$:

$$Q_{m\times m} := \left[\hat{Q}_{m\times n} \quad \hat{Q}_r\right] = \begin{bmatrix} | & | & & | & | & & | \\ \widehat{\mathbf{q}}_1 & \widehat{\mathbf{q}}_2 & \cdots & \widehat{\mathbf{q}}_n & \widehat{\mathbf{q}}_{n+1} & \cdots & \widehat{\mathbf{q}}_m \\ | & | & & | & | & & | \end{bmatrix}, \qquad R_{m\times n} := \begin{bmatrix} \hat{R}_{n\times n} \\ O \end{bmatrix}$$

- **FULL-RANK LEAST-SQUARES PROCEDURE USING FULL QR FACTORIZATION:**

GIVEN: $m \times n$ ($m \geq n$) linear system $A\mathbf{x} = \mathbf{b}$, full column rank A, $\mathbf{b} \notin \text{ColSp}(A)$.

TASK: Find Least-Squares Solution $\mathbf{x}^*$ s.t. $||\mathbf{b} - A\mathbf{x}||_2^2$ is minimized.

(1) Perform Full QR Factorization using CGS-EN: $A = QR$

$$Q_{m\times m} := \left[\hat{Q}_{m\times n} \quad \hat{Q}_r\right], \qquad R_{m\times n} := \begin{bmatrix} \hat{R}_{n\times n} \\ O \end{bmatrix}$$

(2) Find Projection Matrix onto ColSp(A): $\bar{P} = \hat{Q}\hat{Q}^T$

(3) Find best Approximation $\mathbf{b}^* \in \text{ColSp}(A)$ to $\mathbf{b}$: $\mathbf{b}^* = \bar{P}\mathbf{b}$

(4) Find Projection Matrix onto ColSp$(A)^\perp$: $\bar{P}_r = \hat{Q}_r\hat{Q}_r^T$

(5) Minimize square-norm of Residual: $\min\limits_{\mathbf{x}\in\mathbb{R}^n} ||\mathbf{b} - A\mathbf{x}||_2^2 = ||\bar{P}_r\mathbf{b}||_2^2$

(6) Back-solve linear system $\hat{R}\mathbf{x}^* = \hat{Q}^T\mathbf{b}$ for $\mathbf{x}^*$.

- **ORTHOGONAL MATRICES (DEFINITION):** A square matrix Q is **orthogonal** if its columns are orthonormal.
- **ORTHOGONAL MATRICES (PROPERTIES):** The following properties are all equivalent:

(a) Q is an orthogonal matrix

(b) The columns of Q are orthonormal

(c) $Q^TQ = QQ^T = I$

(d) $Q^{-1} = Q^T$

(e) Q^T is an orthogonal matrix

(f) The rows of Q are orthonormal

- **ORTHOGONAL MATRICES (DETERMINANTS):**

(a) Q is orthogonal matrix $\implies \det(Q) = \pm 1$.

(b) $\det(Q) = \pm 1 \;\not\Longrightarrow\; Q$ is orthogonal matrix

- **ORTHOGONAL MATRICES (PRESERVATION):** Consider Euclidean inner product space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_2)$ with...

... inner product $\langle\mathbf{v},\mathbf{w}\rangle_2 := \mathbf{v}^T\mathbf{w}$ and induced norm $||\mathbf{x}||_2 := \langle\mathbf{x},\mathbf{x}\rangle_2$ and induced metric $d_2(\mathbf{v},\mathbf{w}) := ||\mathbf{v}-\mathbf{w}||_2$. Then:

(i) $\langle Q\mathbf{v}, Q\mathbf{w}\rangle_2 = \langle\mathbf{v},\mathbf{w}\rangle_2$, ($ii$) $||Q\mathbf{x}||_2 = ||\mathbf{x}||_2$, ($iii$) $d_2(Q\mathbf{v}, Q\mathbf{w}) = d_2(\mathbf{v},\mathbf{w})$

EX 5.4.1: Find the orthogonal complement of subspace $W = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 8 \end{bmatrix} \right\}$.

EX 5.4.2: Let vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and subspace $W = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \right\} \equiv \text{span}\{\mathbf{q}_1, \mathbf{q}_2\}$. Compute $\text{proj}_W \mathbf{v}$.

EX 5.4.3: Given the inconsistent linear system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(a) Find the least-squares solution $\mathbf{x}^*$ via the normal equations.

(b) Find the minimum square-norm residual of the linear system: $\min_{\mathbf{x} \in \mathbb{R}^2} ||A\mathbf{x} - \mathbf{b}||_2^2$

(c) Find the projection matrix $\bar{P}$ onto the column space of A.

(d) Find the best approximation $\mathbf{b}^* \in \mathrm{ColSp}(A)$ to the RHS vector $\mathbf{b}$.

EX 5.4.4: Given the inconsistent linear system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(a) Perform the Reduced QR Factorization of A using Classical Gram-Schmidt with early normalization.

(b) Find the projection matrix $\bar{P}$ onto the column space of A.

(c) Find the best approximation $\mathbf{b}^* \in \text{ColSp}(A)$ to the RHS vector $\mathbf{b}$.

(d) Find the minimum square-norm residual of the linear system: $\min\limits_{\mathbf{x}\in\mathbb{R}^2} ||A\mathbf{x} - \mathbf{b}||_2^2$

(e) Find the least-squares solution $\mathbf{x}^*$ via the Reduced QR Factorization performed in part (a).

EX 5.4.5: Given the inconsistent linear system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(a) Perform the Full QR Factorization of A using Classical Gram-Schmidt with early normalization.

(b) Find the projection matrix $\bar{P}$ onto ColSp(A).

(c) Find the best approximation $\mathbf{b}^* \in$ ColSp(A) to the RHS vector $\mathbf{b}$.

(d) Find the projection matrix $\bar{P}_r$ onto ColSp$(A)^\perp$.

(e) Find the minimum square-norm residual of the linear system: $\min_{\mathbf{x}\in\mathbb{R}^2} ||A\mathbf{x} - \mathbf{b}||_2^2$

(f) Find the least-squares solution $\mathbf{x}^*$ via the Full QR Factorization performed in part (a).

EX 5.4.6: Perform the Full QR Factorization using Classical Gram-Schmidt with early normalization of $A = \begin{bmatrix} 3 & 0 & 2 \\ -3 & 2 & -1 \\ 3 & -1 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.