EX 6.1.1: Let transformation $T:\mathbb{R}\to\mathbb{R}^2$ s.t. $T(t)=\begin{bmatrix}2t\\0\end{bmatrix}$. Show that T <u>is</u> a linear transformation.

It's sufficient to show that the Superposition Principle (LT5) holds: $[s,t\in\mathbb{R}$ and $\alpha,\beta\in\mathbb{R}]$

$$T(\alpha s+\beta t)=\begin{bmatrix}2(\alpha s+\beta t)\\0\end{bmatrix}=\begin{bmatrix}2\alpha s+2\beta t\\0\end{bmatrix}=\begin{bmatrix}2\alpha s\\0\end{bmatrix}+\begin{bmatrix}2\beta t\\0\end{bmatrix}=\alpha\begin{bmatrix}2s\\0\end{bmatrix}+\beta\begin{bmatrix}2t\\0\end{bmatrix}=\alpha T(s)+\beta T(t)$$

$\therefore$ Since $T(\alpha s+\beta t)=\alpha T(s)+\beta T(t)$, T <u>is</u> a linear transformation.

EX 6.1.2: Let transformation $T:\mathbb{R}\to\mathbb{R}^2$ s.t. $T(t)=\begin{bmatrix}2t\\1\end{bmatrix}$. Show that T is <u>not</u> a linear transformation.

It's sufficient to show that the "Zero Vector Property" (LT3) does not hold in general:

$$T(\vec{\mathbf{0}})=T\left(\begin{bmatrix}0\\0\end{bmatrix}\right)=\begin{bmatrix}2(0)\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}=\vec{\mathbf{0}}\in\mathbb{R}^2$$

$\therefore$ Since $T(\vec{\mathbf{0}})\neq\vec{\mathbf{0}}$, T is <u>not</u> a linear transformation.

EX 6.1.3: Let transformation $T:\mathbb{R}\to\mathbb{R}^2$ s.t. $T(t)=\begin{bmatrix}2t+1\\t\end{bmatrix}$. Show that T is <u>not</u> a linear transformation.

It's sufficient to show that the "Zero Vector Property" (LT3) does not hold in general:

$$T(\vec{\mathbf{0}})=T\left((0,0)^T\right)=(2(0)+1,(0))^T=(1,0)^T\neq(0,0)^T=\vec{\mathbf{0}}\in\mathbb{R}^2$$

$\therefore$ Since $T(\vec{\mathbf{0}})\neq\vec{\mathbf{0}}$, T is <u>not</u> a linear transformation.

EX 6.1.4: Let transformation $T:\mathbb{R}^2\to\mathbb{R}$ s.t. $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=3x_2-x_1$. Show that T <u>is</u> a linear transformation.

Since the rule for T is complicated, show that the Axioms (LT1) & (LT2) hold: $[\mathbf{u}=(u_1,u_2)^T,\mathbf{v}=(v_1,v_2)^T$ and $\alpha\in\mathbb{R}]$

(LT1): $T(\mathbf{u}+\mathbf{v})=T\left(\begin{bmatrix}u_1+v_1\\u_2+v_2\end{bmatrix}\right)=3(u_2+v_2)-(u_1+v_1)=(3u_2-u_1)+(3v_2-v_1)=T(\mathbf{u})+T(\mathbf{v})$

(LT2): $T(\alpha\mathbf{v})=T\left(\begin{bmatrix}\alpha v_1\\\alpha v_2\end{bmatrix}\right)=3(\alpha v_2)-(\alpha v_1)=\alpha(3v_2)-\alpha(v_1)=\alpha(3v_2-v_1)=\alpha T(\mathbf{v})$

$\therefore$ Since $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ and $T(\alpha\mathbf{v})=\alpha T(\mathbf{v})$, T <u>is</u> a linear transformation.

EX 6.1.5: Let transformation $T:\mathbb{R}^2\to\mathbb{R}$ s.t. $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=x_1+4$. Show that T is <u>not</u> a linear transformation.

It's sufficient to show that the "Zero Vector Property" (LT3) does not hold in general:

$$T(\vec{\mathbf{0}})=T\left((0,0)^T\right)=(0)+4=4\neq0=\vec{\mathbf{0}}\in\mathbb{R}$$

$\therefore$ Since $T(\vec{\mathbf{0}})\neq\vec{\mathbf{0}}$, T is <u>not</u> a linear transformation.

EX 6.1.6: Let transformation $T:\mathbb{R}^2\to\mathbb{R}$ s.t. $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=x_1^2$. Show that T is <u>not</u> a linear transformation.

Unfortunately, the "Zero Vector Property" (LT3) does indeed hold:

$$T(\vec{\mathbf{0}})=T\left((0,0)^T\right)=(0)^2=0=\vec{\mathbf{0}}\in\mathbb{R}\implies T(\vec{\mathbf{0}})=\vec{\mathbf{0}}$$

Hence, it's easiest to show that the SM Axiom (LT2) does not hold for all $\alpha\in\mathbb{R}$:

$$T(\alpha\mathbf{v})=T\left((\alpha v_1,\alpha v_2)^T\right)=(\alpha v_1)^2=\alpha^2v_1^2\neq\alpha v_1^2=\alpha T(\mathbf{v})\quad\text{for}\quad\alpha\neq1$$

$\therefore$ Since $T(\alpha\mathbf{v})\neq\alpha T(\mathbf{v})$ in general, T is <u>not</u> a linear transformation.

EX 6.1.7: Let transformation $T:\mathbb{R}^2\to\mathbb{R}$ s.t. $T(\mathbf{x})=||\mathbf{x}||$. Show that T is <u>not</u> a linear transformation.

Alas, the "Zero Vector Property" does indeed hold: $T(\vec{\mathbf{0}})=||(0,0)^T||=\sqrt{(0)^2+(0)^2}=0=\vec{\mathbf{0}}\in\mathbb{R}\implies T(\vec{\mathbf{0}})=\vec{\mathbf{0}}$

Hence, it's easiest to show that the SM Axiom (LT2) does not hold for all $\alpha\in\mathbb{R}$:

$$T(\alpha\mathbf{v})=||\alpha\mathbf{v}||\overset{NM3}{=}|\alpha|||\mathbf{v}||\neq\alpha||\mathbf{v}||=\alpha T(\mathbf{v})\quad\text{for}\quad\alpha<0$$

$\therefore$ Since $T(\alpha\mathbf{v})\neq\alpha T(\mathbf{v})$ in general, T is <u>not</u> a linear transformation.

EX 6.1.8: Let transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ s.t. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)=\begin{bmatrix} x_1-x_2 \\ x_1 \end{bmatrix}$. Show that T is a linear transformation.

It's sufficient to show that the Superposition Principle (LT5) holds: $[\mathbf{u}=(u_1,u_2)^T$ and $\mathbf{v}=(v_1,v_2)^T$ and $\alpha,\beta\in\mathbb{R}]$

$$\begin{aligned} T(\alpha\mathbf{u}+\beta\mathbf{v}) &= T\left(\begin{bmatrix} \alpha u_1+\beta v_1 \\ \alpha u_2+\beta v_2 \end{bmatrix}\right)=\begin{bmatrix} (\alpha u_1+\beta v_1)-(\alpha u_2+\beta v_2) \\ (\alpha u_1+\beta v_1) \end{bmatrix}=\begin{bmatrix} (\alpha u_1-\alpha u_2)+(\beta v_1-\beta v_2) \\ (\alpha u_1)+(\beta v_1) \end{bmatrix} \\ &= \begin{bmatrix} \alpha u_1-\alpha u_2 \\ \alpha u_1 \end{bmatrix}+\begin{bmatrix} \beta v_1-\beta v_2 \\ \beta v_1 \end{bmatrix}=\alpha\begin{bmatrix} u_1-u_2 \\ u_1 \end{bmatrix}+\beta\begin{bmatrix} v_1-v_2 \\ v_1 \end{bmatrix}=\alpha T(\mathbf{u})+\beta T(\mathbf{v}) \end{aligned}$$

$\therefore$ Since $T(\alpha\mathbf{u}+\beta\mathbf{v})=\alpha T(\mathbf{u})+\beta T(\mathbf{v})$, T is a linear transformation.

EX 6.1.9: Let transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ s.t. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)=\begin{bmatrix} x_1-x_2 \\ x_1-2 \end{bmatrix}$. Show that T is not a linear transformation.

It's sufficient to show that the "Zero Vector Property" (LT3) does not hold in general:

$$T(\vec{\mathbf{0}})=T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)=\begin{bmatrix} (0)-(0) \\ (0)-2 \end{bmatrix}=\begin{bmatrix} 0 \\ -2 \end{bmatrix}\neq\begin{bmatrix} 0 \\ 0 \end{bmatrix}=\vec{\mathbf{0}}\in\mathbb{R}^2$$

$\therefore$ Since $T(\vec{\mathbf{0}})\neq\vec{\mathbf{0}}$, T is not a linear transformation.

EX 6.1.10: Let transformation $T:\mathbb{R}^{3\times2}\to\mathbb{R}^{2\times3}$ s.t. $T(A)=A^T$. Show that T is a linear transformation.

It's sufficient to show that the Superposition Principle (LT5) holds: $[A,B\in\mathbb{R}^{3\times2}$ and $\alpha,\beta\in\mathbb{R}]$

$$T(\alpha A+\beta B)=(\alpha A+\beta B)^T\overset{T2}{=}(\alpha A)^T+(\beta B)^T\overset{T3}{=}\alpha(A^T)+\beta(B^T)=\alpha T(A)+\beta T(B)$$

$\therefore$ Since $T(\alpha A+\beta B)=\alpha T(A)+\beta T(B)$, T is a linear transformation.

EX 6.1.11: Let transformation $T:\mathbb{R}^{2\times2}\to\mathbb{R}^{2\times2}$ s.t. $T(A)=AX-XA$ where X is an arbitrary 2×2 matrix.

Show that T is a linear transformation.

Since the rule for T is complicated, show that the Axioms (LT1) & (LT2) hold: $[A,B\in\mathbb{R}^{2\times2}$ and $\alpha\in\mathbb{R}]$

(LT1): $T(A+B)=(A+B)X-X(A+B)\overset{M3/M4}{=}AX+BX-XA-XB=(AX-XA)+(BX-XB)=T(A)+T(B)$

(LT2): $T(\alpha A)=(\alpha A)X-X(\alpha A)\overset{M2}{=}\alpha(AX)-\alpha(XA)=\alpha(AX-XA)=\alpha T(A)$

$\therefore$ Since $T(A+B)=T(A)+T(B)$ and $T(\alpha A)=\alpha T(A)$, T is a linear transformation.

EX 6.1.12: Let transformation $T:\mathbb{R}^{2\times2}\to\mathbb{R}^{2\times2}$ s.t. $T(A)=A^{-1}$. Show that T is not a linear transformation.

Since $O_{2\times2}$ is not invertible, $O_{2\times2}\notin\text{CoDomain}(T)\implies$ "Zero Vector Property" (LT3) does not hold.

Alternatively, show that the SM Axiom (LT2) does not hold for all $\alpha\in\mathbb{R}$:

(LT2): $T(\alpha A)=(\alpha A)^{-1}\overset{I3}{=}\frac{1}{\alpha}A^{-1}=\frac{1}{\alpha}T(A)\neq\alpha T(A)$ when $\alpha\neq1$

$\therefore$ Since $T(\alpha A)\neq\alpha T(A)$ in general, T is not a linear transformation.

EX 6.1.13: Let transformation $T:P_3\to P_5$ s.t. $T(p)=x^2p(x)$. Show that T is a linear transformation.

It's sufficient to show that the Superposition Principle (LT5) holds: $[p,q\in P_3$ and $\alpha,\beta\in\mathbb{R}]$

$$T(\alpha p+\beta q)=x^2[\alpha p(x)+\beta q(x)]=\alpha x^2p(x)+\beta x^2q(x)=\alpha[x^2p(x)]+\beta[x^2q(x)]=\alpha T(p)+\beta T(q)$$

$\therefore$ Since $T(\alpha p+\beta q)=\alpha T(p)+\beta T(q)$, T is a linear transformation.

EX 6.1.14: Let transformation $T:P_3\to P_1$ s.t. $T(p)=p''(x)$. Show that T is a linear transformation.

$$T(\alpha p+\beta q)=[\alpha p(x)+\beta q(x)]''\overset{CALCULUS}{=}[\alpha p(x)]''+[\beta q(x)]''\overset{CALCULUS}{=}\alpha p''(x)+\beta q''(x)=\alpha T(p)+\beta T(q)$$

$\therefore$ Since $T(\alpha p+\beta q)=\alpha T(p)+\beta T(q)$, T is a linear transformation.

EX 6.1.15: Let transformation $T:P_3\to P_3$ s.t. $T(p)=x^3+p(x)$. Show that T is not a linear transformation.

It's sufficient to show that the "Zero Vector Property" (LT3) does not hold in general:

$$T(\vec{\mathbf{0}})=T(z(x))=T(0x^3+0x^2+0x+0)=x^3+0=x^3\neq0x^3+0x^2+0x+0=\vec{\mathbf{0}}\in P_3$$

$\therefore$ Since $T(\vec{\mathbf{0}})\neq\vec{\mathbf{0}}$, T is not a linear transformation.

EX 6.1.16: Let transformation $T : C[0,1] \to \mathbb{R}$ s.t. $T(f) = \int_0^1 x f(x)\, dx$. Show that T is a linear transformation.

It's sufficient to show that the Superposition Principle (LT5) holds: $[f, g \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}]$

$$\begin{aligned} T(\alpha f + \beta g) &= \int_0^1 x[\alpha f(x) + \beta g(x)]\, dx = \int_0^1 [\alpha x f(x) + \beta x g(x)]\, dx = \int_0^1 \alpha x f(x)\, dx + \int_0^1 \beta x g(x)\, dx \\ &= \alpha \int_0^1 x f(x)\, dx + \beta \int_0^1 x g(x)\, dx = \alpha T(f) + \beta T(g) \end{aligned}$$

$\therefore$ Since $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$, T is a linear transformation.

EX 6.1.17: Let transformation $T : C[0,1] \to \mathbb{R}$ s.t. $T(f) = \int_0^1 [x + f(x)]\, dx$. Show that T is not a linear transformation.

It's sufficient to show that the "Zero Vector Property" (LT3) does not hold in general: $[z(x) = 0]$

$$T(\vec{\mathbf{0}}) = T(z(x)) = \int_0^1 [x + z(x)]\, dx = \int_0^1 [x + (0)]\, dx = \int_0^1 x\, dx = \left[\frac{1}{2}x^2\right]_{x=0}^{x=1} \overset{FTC}{=} \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2} \neq 0 = \vec{\mathbf{0}} \in \mathbb{R}$$

$\therefore$ Since $T(\vec{\mathbf{0}}) \neq \vec{\mathbf{0}}$, T is not a linear transformation.

EX 6.1.18: Let linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

(a) Compute $L\left(\begin{bmatrix} -4 \\ 8 \end{bmatrix}\right)$.

The key is to write vector $(-4, 8)^T$ as a linear combination of the two given input vectors $(1, 0)^T$ and $(0, 1)^T$ and then use the Superposition Principle (LT5):

$$\begin{aligned} L\left(\begin{bmatrix} -4 \\ 8 \end{bmatrix}\right) &= L\left((-4)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (8)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \overset{LT5}{=} (-4)L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + (8)L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = (-4)\begin{bmatrix} -3 \\ 2 \end{bmatrix} + (8)\begin{bmatrix} 1 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 12 \\ -8 \end{bmatrix} + \begin{bmatrix} 8 \\ 40 \end{bmatrix} = \boxed{\begin{bmatrix} 20 \\ 32 \end{bmatrix}} \end{aligned}$$

(b) Compute $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$, where $x_1, x_2 \in \mathbb{R}$.

$$\begin{aligned} L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= L\left((x_1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (x_2)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \overset{LT5}{=} (x_1)L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + (x_2)L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = (x_1)\begin{bmatrix} -3 \\ 2 \end{bmatrix} + (x_2)\begin{bmatrix} 1 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -3x_1 \\ 2x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 5x_2 \end{bmatrix} = \boxed{\begin{bmatrix} -3x_1 + x_2 \\ 2x_1 + 5x_2 \end{bmatrix}} \end{aligned}$$

EX 6.1.19: Let linear transformation $L : P_1 \to P_1$ s.t. $L(1) = x + 5$ and $L(x) = 2 - 3x$.

(a) Compute $L(4x - 3)$.

Write linear polynomial $4x - 3$ as a linear combination of the two given input linears 1 and x and then use the Superposition Principle (LT5):

$$L(4x - 3) = L[4(x) - 3(1)] \overset{LT5}{=} 4L(x) - 3L(1) = 4(2 - 3x) - 3(x + 5) = 8 - 12x - 3x - 15 = \boxed{-15x - 7}$$

(b) Compute $L(ax + b)$, where $a, b \in \mathbb{R}$.

$$L(ax + b) = L[a(x) + b(1)] \overset{LT5}{=} aL(x) + bL(1) = a(2 - 3x) + b(x + 5) = 2a - 3ax + bx + 5b = \boxed{(b - 3a)x + (2a + 5b)}$$