LINEAR TRANSFORMATIONS [LARSON 6.1]

- **COMMON VECTOR SPACES (REVIEW):**

$$
\begin{aligned}
\mathbb{R} &\equiv \text{Set of all real numbers (scalars)}\\
\mathbb{R}^2 &\equiv \text{Set of all ordered pairs (2-wide vectors)}\\
\mathbb{R}^3 &\equiv \text{Set of all ordered triples (3-wide vectors)}\\
\mathbb{R}^n &\equiv \text{Set of all ordered } n\text{-tuples (} n\text{-wide vectors)}\\
\mathbb{R}^{m\times n} &\equiv \text{Set of all } m \times n \text{ matrices}\\
\mathbb{R}^{n\times n} &\equiv \text{Set of all } n \times n \text{ square matrices}\\
P_n &\equiv \text{Set of all polynomials of degree } n \text{ or less}\\
C[a,b] &\equiv \text{Set of all continuous functions on } [a,b]\\
C^1[a,b] &\equiv \text{Set of all differentiable functions on } [a,b]\\
C^2[a,b] &\equiv \text{Set of all twice-differentiable fcns on } [a,b]\\
C(\mathbb{R}) &\equiv \text{Set of all everywhere-continuous functions}\\
C^1(\mathbb{R}) &\equiv \text{Set of all everywhere-differentiable functions}\\
C^2(\mathbb{R}) &\equiv \text{Set of all everywhere-twice-differentiable fcns}\\
C^\infty(\mathbb{R}) &\equiv \text{Set of all everywhere-infinitely-differentiable fcns}
\end{aligned}
$$

- **TRANSFORMATION (DEFINITION):** A **transformation** T is a function between vector spaces V, W.

  The **signature** of transformation T from vector spaces V into W is written: $T : V \to W$

  V is the **domain** of T and W is the **codomain** of T. $T(\mathbf{v}) \in W$ is called the **image** of vector $\mathbf{v} \in V$.

  The set of all images of all vectors in V is the **range** of T: $\text{Range}(T) := \{T(\mathbf{v}) : \mathbf{v} \in V\} \subseteq W$

- **LINEAR TRANSFORMATION (DEFINITION):** Let V, W be vector spaces and transformation $T : V \to W$.

  T is a **linear transformation** if the following all hold $\forall \mathbf{u}, \mathbf{v} \in V$ and $\forall \alpha \in \mathbb{R}$:

(LT1)	$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$	Preservation of Vector Addition
(LT2)	$T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$	Preservation of Scalar Mult.

  i.e. Linear transformations preserve vector addition & scalar multiplication.

  REMARK: If a transformation is linear a priori, then the linear transformation will be denoted by L instead of T.

- **LINEAR TRANSFORMATION (PROPERTIES):** Let $L : V \to W$ be a linear transformation.

  Then the following all hold $\forall \mathbf{u}, \mathbf{v} \in V$ and $\forall \alpha, \beta \in \mathbb{R}$:

(LT3)	$L(\vec{\mathbf{0}}) = \vec{\mathbf{0}}$	L maps $\vec{\mathbf{0}} \in V$ to $\vec{\mathbf{0}} \in W$
(LT4)	$L(\mathbf{u}-\mathbf{v}) = L(\mathbf{u}) - T(\mathbf{v})$	Preservation of Vector Subtraction
(LT5)	$L(\alpha\mathbf{u}+\beta\mathbf{v}) = \alpha L(\mathbf{u}) + \beta L(\mathbf{v})$	Superposition Principle

  Establishing (LT5) is sufficient when showing T is a linear transformation.

  (LT3) may be helpful when showing T is not a linear transformation.

- **TWO SPECIAL LINEAR TRANSFORMATIONS:** Let $L : V \to W$ be a linear transformation. Then:

(ZT)	L is the **zero transformation**	if $L(\mathbf{v}) = \vec{\mathbf{0}}$ $\forall \mathbf{v} \in V$
(IT)	L is the **identity transformation**	if $L(\mathbf{v}) = \mathbf{v}$ $\forall \mathbf{v} \in V$

- **LINEAR TRANSFORMATION GIVEN BY A MATRIX:** Let A be an $m \times n$ matrix.

  Then $T(\mathbf{v}) = A\mathbf{v}$ is a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$: $T : \mathbb{R}^n \to \mathbb{R}^m$

EXAMPLES OF TRANSFORMATIONS [LARSON 6.1]

SIGNATURE	EXAMPLE	SPECIAL NAME
$T : \mathbb{R} \to \mathbb{R}$	$T(x) = x^2 + 1$	(Scalar) Function
$T : \mathbb{R} \to \mathbb{R}^n$	$T(t) = \begin{bmatrix} 1 - t \\ \sqrt{t} \end{bmatrix}$	Vector Function
$T : \mathbb{R}^m \to \mathbb{R}$	$T(\mathbf{x}) = 2\mathbf{x}^T\mathbf{x}$	Scalar Field
$T : \mathbb{R}^m \to \mathbb{R}$	$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1^3 - x_1 x_2$	Scalar Field
$T : \mathbb{R}^m \to \mathbb{R}^n$	$T(\mathbf{x}) = A\mathbf{x} \qquad (A \in \mathbb{R}^{3\times 4})$	Vector Field
$T : \mathbb{R}^m \to \mathbb{R}^n$	$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \sin x_3 \\ \ln x_2 - x_1^2 \end{bmatrix}$	Vector Field
$T : \mathbb{R} \to \mathbb{R}^{m\times n}$	$T(t) = t^2 I \qquad (I \text{ is } 3 \times 3 \text{ identity matrix})$	Matrix Function
$T : \mathbb{R} \to \mathbb{R}^{m\times n}$	$T(t) = \begin{bmatrix} 8 & (1 - t^4) \\ \cos(\pi t) & 5e^t \end{bmatrix}$	Matrix Function
$T : \mathbb{R}^{n\times n} \to \mathbb{R}$	$T(A) = \det(A) + 7$	????
$T : \mathbb{R}^{m\times n} \to \mathbb{R}$	$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right) = a_{11}a_{21} - a_{32}^2$	????
$T : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$	$T(A) = I - 3A$	????
$T : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$	$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right) = \begin{bmatrix} a_{23}^2 & 0 \\ a_{13}a_{11} & a_{13} \\ \sqrt[3]{a_{11}} & 6a_{22} \end{bmatrix}$	????
$T : P_n \to \mathbb{R}$	$T(p) = 3p(1) - 2p(-1) + 4$	????
$T : P_2 \to \mathbb{R}$	$T(ax^2 + bx + c) = 3a - b^4 + c^{2/3}$	????
$T : P_{n+2} \to P_n$	$T(p) = p''(x)$	????
$T : P_n \to P_{n+1}$	$T(p) = \int p(x)\, dx$	????
$T : C^2(\mathbb{R}) \to \mathbb{R}$	$T(f) = 3f''(2) - 2f'(1) + f(0)$	Functional
$T : C[0,1] \to \mathbb{R}$	$T(f) = \int_0^1 x^2 f(x)\, dx$	Functional

EX 6.1.1: Let transformation $T : \mathbb{R} \to \mathbb{R}^2$ s.t. $T(t) = \begin{bmatrix} 2t \\ 0 \end{bmatrix}$. Show that T is a linear transformation.

EX 6.1.2: Let transformation $T : \mathbb{R} \to \mathbb{R}^2$ s.t. $T(t) = \begin{bmatrix} 2t \\ 1 \end{bmatrix}$. Show that T is not a linear transformation.

EX 6.1.3: Let transformation $T : \mathbb{R} \to \mathbb{R}^2$ s.t. $T(t) = \begin{bmatrix} 2t+1 \\ t \end{bmatrix}$. Show that T is not a linear transformation.

EX 6.1.4: Let transformation $T : \mathbb{R}^2 \to \mathbb{R}$ s.t. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = 3x_2 - x_1$. Show that T is a linear transformation.

EX 6.1.5: Let transformation $T : \mathbb{R}^2 \to \mathbb{R}$ s.t. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + 4$. Show that T is not a linear transformation.

EX 6.1.6: Let transformation $T : \mathbb{R}^2 \to \mathbb{R}$ s.t. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1^2$. Show that T is not a linear transformation.

EX 6.1.7: Let transformation $T : \mathbb{R}^2 \to \mathbb{R}$ s.t. $T(\mathbf{x}) = ||\mathbf{x}||$. Show that T is not a linear transformation.

EX 6.1.8: Let transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 \end{bmatrix}$. Show that T is a linear transformation.

EX 6.1.9: Let transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 - 2 \end{bmatrix}$. Show that T is not a linear transformation.

EX 6.1.10: Let transformation $T : \mathbb{R}^{3\times 2} \to \mathbb{R}^{2\times 3}$ s.t. $T(A) = A^T$. Show that T is a linear transformation.

EX 6.1.11: Let transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ s.t. $T(A) = AX - XA$ where X is an arbitrary 2×2 matrix. Show that T is a linear transformation.

EX 6.1.12: Let transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ s.t. $T(A) = A^{-1}$. Show that T is not a linear transformation.

EX 6.1.13: Let transformation $T : P_3 \to P_5$ s.t. $T(p) = x^2 p(x)$. Show that T is a linear transformation.

EX 6.1.14: Let transformation $T : P_3 \to P_1$ s.t. $T(p) = p''(x)$. Show that T is a linear transformation.

EX 6.1.15: Let transformation $T : P_3 \to P_3$ s.t. $T(p) = x^3 + p(x)$. Show that T is not a linear transformation.

EX 6.1.16: Let transformation $T : C[0,1] \to \mathbb{R}$ s.t. $T(f) = \int_0^1 x f(x)\, dx$. Show that T is a linear transformation.

EX 6.1.17: Let transformation $T : C[0,1] \to \mathbb{R}$ s.t. $T(f) = \int_0^1 [x + f(x)]\, dx$. Show that T is not a linear transformation.

EX 6.1.18: Let linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

(a) Compute $L\left(\begin{bmatrix} -4 \\ 8 \end{bmatrix}\right)$.

(b) Compute $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$, where $x_1, x_2 \in \mathbb{R}$.

EX 6.1.19: Let linear transformation $L : P_1 \to P_1$ s.t. $L(1) = x + 5$ and $L(x) = 2 - 3x$.

(a) Compute $L(4x - 3)$.

(b) Compute $L(ax + b)$, where $a, b \in \mathbb{R}$.