EX 6.2.2: Let linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ s.t. $L(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 2 & 3 \end{bmatrix}$.

(a) Find the kernel of L, $\ker(L)$.

Realize that $\ker(L) = \text{NulSp}(A)$, so find the null space of matrix A:

$$[A \mid \vec{\mathbf{0}}] = \left[\begin{array}{ccc|c} \boxed{1} & 3 & 3 & 0 \\ 3 & 2 & 3 & 0 \end{array}\right] \xrightarrow{(-3)R_1+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 3 & 0 \\ 0 & -7 & -6 & 0 \end{array}\right] \xrightarrow{(-\frac{1}{7})R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 3 & 0 \\ 0 & \boxed{1} & 6/7 & 0 \end{array}\right]$$

$$\xrightarrow{(-3)R_2+R_1\to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 3/7 & 0 \\ 0 & \boxed{1} & 6/7 & 0 \end{array}\right] = [\text{RREF}(A) \mid \vec{\mathbf{0}}]$$

Column 3 of $\text{RREF}(A)$ has no pivot, so x_3 is a free variable. Hence, let $x_3 = 7t$

Interpreting the rows of $\text{RREF}(A)$ yields: $\begin{cases} x_1 + \frac{3}{7}x_3 = 0 \\ x_2 + \frac{6}{7}x_3 = 0 \end{cases} \implies \begin{cases} x_1 = -\frac{3}{7}(7t) = -3t \\ x_2 = -\frac{6}{7}(7t) = -6t \end{cases}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3t \\ -6t \\ 7t \end{bmatrix} = t\begin{bmatrix} -3 \\ -6 \\ 7 \end{bmatrix} \implies \boxed{\ker(L) = \text{span}\left\{\begin{bmatrix} -3 \\ -6 \\ 7 \end{bmatrix}\right\}}$$

(b) Find the nullity of L, $\text{nullity}(L)$.

$\text{nullity}(L) = \dim[\ker(L)] = [\#\text{ non-pivot columns of RREF}(A)] = \boxed{1}$

(c) Find the range of L, $\text{range}(L)$.

$\text{range}(L) = \text{ColSp}(A) = \text{span}\{\text{pivot columns of } A\} = \boxed{\text{span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}}$

(d) Find the rank of L, $\text{rank}(L)$.

$\text{rank}(L) = \dim[\text{range}(L)] = [\#\text{ pivot columns of RREF}(A)] = \boxed{2}$

(e) Is L one-to-one? (Justify answer.)

No, since $\ker(L) \neq \{\vec{\mathbf{0}}\}$, meaning more than one vector in the domain gets mapped to the zero vector in the codomain.

——— OR ———

No, since $\text{nullity}(L) > 0$, meaning more than one vector in the domain gets mapped to the zero vector in the codomain.

(f) Is L onto? (Justify answer.)

Yes, since $\text{rank}(L) = 2 = \dim(\mathbb{R}^2) = \dim[\text{codomain}(L)]$, meaning the range of L is the entire codomain of L.

EX 6.2.3: Let linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $L(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 2 & 3 \\ -3 & 3 & 2 \end{bmatrix}$.

(a) Find the kernel of L, $\ker(L)$.

Realize that $\ker(L) = \text{NulSp}(A)$, so find the null space of matrix A:

$$[A \mid \vec{\mathbf{0}}] = \left[\begin{array}{ccc|c} \boxed{1} & 3 & 3 & 0 \\ 3 & 2 & 3 & 0 \\ -3 & 3 & 2 & 0 \end{array}\right] \xrightarrow[3R_1+R_3\to R_3]{(-3)R_1+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 3 & 0 \\ 0 & -7 & -6 & 0 \\ 0 & 12 & 11 & 0 \end{array}\right] \xrightarrow[7R_3\to R_3]{(-12)R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 3 & 0 \\ 0 & 84 & 72 & 0 \\ 0 & 84 & 77 & 0 \end{array}\right]$$

$$\xrightarrow{(-1)R_2+R_3\to R_3} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 3 & 0 \\ 0 & 84 & 72 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right] \xrightarrow[(\frac{1}{5})R_3\to R_3]{(\frac{1}{12})R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 3 & 0 \\ 0 & 7 & 6 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right] \xrightarrow[(-3)R_3+R_1\to R_1]{(-6)R_3+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

$$\xrightarrow{(\frac{1}{7})R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right] \xrightarrow{(-3)R_2+R_1\to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right] = [\text{RREF}(A) \mid \vec{\mathbf{0}}]$$

Since every column of $\text{RREF}(A)$ has a pivot, there are no free variables $\implies x_1, x_2, x_3$ are all fixed variables

Interpreting the rows of $\text{RREF}(A)$ yields: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \boxed{\ker(L) = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}} = \text{span}\left\{\vec{\mathbf{0}}\right\} = \left\{\vec{\mathbf{0}}\right\}$

(b) Find the nullity of L, $\text{nullity}(L)$.

$\text{nullity}(L) = \dim[\ker(L)] = [\#\text{ non-pivot columns of RREF}(A)] = \boxed{0}$

(c) Find the range of L, $\text{range}(L)$.

$\text{range}(L) = \text{ColSp}(A) = \text{span}\{\text{pivot columns of } A\} = \boxed{\text{span}\left\{\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}\right\}}$

(d) Find the rank of L, $\text{rank}(L)$.

$\text{rank}(L) = \dim[\text{range}(L)] = [\#\text{ pivot columns of RREF}(A)] = \boxed{3}$

(e) Is L one-to-one? (Justify answer.)

Yes, since $\ker(L) = \{\vec{\mathbf{0}}\}$, meaning only the zero vector in the domain gets mapped to the zero vector in the codomain.

——— OR ———

Yes, since $\text{nullity}(L) = 0$, meaning only the zero vector in the domain gets mapped to the zero vector in the codomain.

(f) Is L onto? (Justify answer.)

Yes, since $\text{rank}(L) = 3 = \dim(\mathbb{R}^3) = \dim[\text{codomain}(L)]$, meaning the range of L is the entire codomain of L.