LINEAR TRANSFORMATIONS: KERNEL, RANGE, 1-1, ONTO [LARSON 6.2]

- **KERNEL OF A LINEAR TRANSFORMATION (DEFINITION):** Let $L : V \to W$ be a linear transformation.

  Then the **kernel** of L is defined to be: $\ker(L) := \{\mathbf{v} \in V : L(\mathbf{v}) = \vec{\mathbf{0}}\} \subseteq V$

  i.e. The kernel of L is the set of all vectors in V that are mapped by L to $\vec{\mathbf{0}}$.

- **THE KERNEL IS A SUBSPACE:** Let $L : V \to W$ be a linear transformation. Then, $\ker(L)$ is a subspace of V.

- **RANGE OF A LINEAR TRANSFORMATION (DEFINITION):** Let $L : V \to W$ be a linear transformation.

  Then the **range** of L is defined to be: $\text{range}(L) := \{L(\mathbf{v}) \in W : \mathbf{v} \in V\}$

- **THE RANGE IS A SUBSPACE:** Let $L : V \to W$ be a linear transformation. Then, $\text{range}(L)$ is a subspace of W.

- **KER(L) & RANGE(L) IN TERMS OF REPRESENTATIVE MATRIX:**

$$\text{Let } L : \mathbb{R}^n \to \mathbb{R}^m \text{ such that } L(\mathbf{x}) = A\mathbf{x}. \quad (\text{where } A \in \mathbb{R}^{m\times n}) \quad \text{Then:}$$

$$\ker(L) = \text{NulSp}(A) \qquad \text{range}(L) = \text{ColSp}(A)$$

- **RANK & NULLITY OF A LINEAR TRANSFORMATION:**

  (i) Let A be an $m \times n$ matrix. Then:

$$\begin{aligned} \text{rank}(A) &= \dim[\text{ColSp}(A)] = (\#\text{ of pivot columns of RREF}(A)) \\ \text{nullity}(A) &= \dim[\text{NulSp}(A)] = (\#\text{ of non-pivot columns of RREF}(A)) \end{aligned}$$

  (ii) Let $L : V \to W$ be a linear transformation. Then:

$$\text{rank}(L) = \dim[\text{range}(L)] \qquad \text{nullity}(L) = \dim[\ker(L)]$$

- **PREIMAGE OF A VECTOR (DEFINITION):** Let $T : V \to W$ be a transformation.

  Then the **preimage** of vector $\mathbf{w} \in W$ is:

$$\text{Preimage}(\mathbf{w}) := \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{w}\} \subseteq V$$

- **1-1 TRANSFORMATION (DEFINITION):** Let $T : V \to W$ be a transformation. Then:

$$T \text{ is } \mathbf{1\text{-}1} \ (\textbf{one-to-one}) \iff \forall \mathbf{u}, \mathbf{v} \in V, \ T(\mathbf{u}) = T(\mathbf{v}) \implies \mathbf{u} = \mathbf{v}.$$

- **1-1 LINEAR TRANSFORMATION (DEFINITION):** Let $L : V \to W$ be a linear transformation. Then:

$$L \text{ is } \mathbf{1\text{-}1} \ (\textbf{one-to-one}) \iff \ker(L) = \{\vec{\mathbf{0}}\}.$$

- **ONTO TRANSFORMATION (DEFINITION):** Let $T : V \to W$ be a transformation. Then:

$$T \text{ is } \textbf{onto} \iff \text{range}(T) = W.$$

- **ONTO LINEAR TRANSFORMATION (DEFINITION):**

  Let $L : V \to W$ be a linear transformation such that W is finite-dimensional. Then:

$$L \text{ is } \textbf{onto} \iff \text{rank}(L) = \dim(W).$$

- **ISOMORPHISM (DEFINITION):** Let $L : V \to W$ be a linear transformation. Then:

  L is called an **isomorphism** if L is 1-1 and onto.

  Moreover, vector spaces V, W are **isomorphic** (to each other) if there exists an isomorphism from V to W.

- **FINITE-DIMENSIONAL VECTOR SPACES & ISOMORPHISMS:**

  Let V, W be finite-dimensional vector spaces. Then:

$$V \ \& \ W \text{ are isomorphic} \iff \dim(V) = \dim(W)$$

EX 6.2.1: Let linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^3$ s.t. $L(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ -3 & 3 \end{bmatrix}$.

(a) Find the kernel of L, $\ker(L)$.

(b) Find the nullity of L, $\text{nullity}(L)$.

(c) Find the range of L, $\text{range}(L)$.

(d) Find the rank of L, $\text{rank}(L)$.

(e) Is L one-to-one? (Justify answer.)

(f) Is L onto? (Justify answer.)

EX 6.2.2: Let linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ s.t. $L(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 2 & 3 \end{bmatrix}$.

(a) Find the kernel of L, $\ker(L)$.

(b) Find the nullity of L, $\text{nullity}(L)$.

(c) Find the range of L, $\text{range}(L)$.

(d) Find the rank of L, $\text{rank}(L)$.

(e) Is L one-to-one? (Justify answer.)

(f) Is L onto? (Justify answer.)

EX 6.2.3: Let linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $L(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 2 & 3 \\ -3 & 3 & 2 \end{bmatrix}$.

(a) Find the kernel of L, $\ker(L)$.

(b) Find the nullity of L, $\text{nullity}(L)$.

(c) Find the range of L, $\text{range}(L)$.

(d) Find the rank of L, $\text{rank}(L)$.

(e) Is L one-to-one? (Justify answer.)

(f) Is L onto? (Justify answer.)