LINEAR TRANSFORMATIONS: STANDARD MATRIX I [LARSON 6.3]

- **STANDARD BASES OF $\mathbb{R}^n$ (REVIEW):**

NOTATION: A standard basis is denoted by $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$, where $\mathbf{e}_j$ is the j^{th} standard basis vector.

With a standard basis, the coefficients in the linear combination are simply the entries of the vector:

$$\star \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{e}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{e}_2} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

$$\star \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{e}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{e}_2} + x_3 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{e}_3} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

- **STANDARD MATRIX FOR A LINEAR TRANSFORMATION (DEFINITION):**

Let linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ s.t. $L(\mathbf{x}) = A\mathbf{x}$ $\forall \mathbf{x} \in \mathbb{R}^n$, where A is $m \times n$ matrix.

Then A is called the **standard matrix** for linear transformation L.

- **FINDING STANDARD MATRIX – EASY CASE:**

GIVEN: Linear Transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ s.t. L is explicitly defined.

TASK: Find Standard Matrix $A \in \mathbb{R}^{m\times n}$ s.t. $L(\mathbf{x}) = A\mathbf{x}$

$$(1) \quad A = \begin{bmatrix} | & | & & | \\ L(\mathbf{e}_1) & L(\mathbf{e}_2) & \cdots & L(\mathbf{e}_n) \\ | & | & & | \end{bmatrix}$$

i.e. The columns of A are the images of the **standard basis vectors** for $\mathbb{R}^n$.

- **COMPOSITION OF LINEAR TRANSFORMATIONS:**

Let linear transformation $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ s.t. $L_1(\mathbf{x}) = A_1\mathbf{x}$.

Let linear transformation $L_2 : \mathbb{R}^m \to \mathbb{R}^p$ s.t. $L_2(\mathbf{x}) = A_2\mathbf{x}$.

Then the **composition** $L_2 \circ L_1$ is defined by $(L_2 \circ L_1)(\mathbf{x}) := L_2[L_1(\mathbf{x})]$.

Moreover, the composition $L_2 \circ L_1$ is a linear transformation and $(L_2 \circ L_1)(\mathbf{x}) = (A_2A_1)\mathbf{x}$

- **INVERSE LINEAR TRANSFORMATION (DEFINITION):**

Let linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ have identical domain & codomain.

Then linear transformation $L^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is the **inverse** of L if

$$(L^{-1} \circ L)(\mathbf{x}) = \mathbf{x} \text{ and } (L \circ L^{-1})(\mathbf{x}) = \mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n$$

L is called **invertible** if its inverse L^{-1} exists.

- **INVERSE LINEAR TRANSFORMATION IN TERMS OF STANDARD MATRIX:**

Let linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $L(\mathbf{x}) = A\mathbf{x}$.

If L is invertible, then its inverse $L^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that $L^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$

LINEAR TRANSFORMATIONS: STANDARD MATRIX II [LARSON 6.3]

- **FINDING STANDARD MATRIX – HARD CASE:**

GIVEN: Linear Transformation L s.t. images $L(\mathbf{b}_1), \ldots, L(\mathbf{b}_n)$ are provided.

where $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a non-standard ordered basis for $\mathbb{R}^n$.

TASK: Find Standard Matrix $A \in \mathbb{R}^{m \times n}$ s.t. $L(\mathbf{x}) = A\mathbf{x}$

(1) $$\left[\begin{array}{ccc|c} \text{——} & \mathbf{b}_1 & \text{——} & L(\mathbf{b}_1) \\ & \vdots & & \vdots \\ \text{——} & \mathbf{b}_n & \text{——} & L(\mathbf{b}_n) \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{c|c} & L(\mathbf{e}_1) \\ I & \vdots \\ & L(\mathbf{e}_n) \end{array}\right]$$

NOTE: Write non-standard basis vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ as **row vectors** (not column vectors.)

NOTE: Write the images $L(\mathbf{b}_1), \ldots, L(\mathbf{b}_n)$ as **transposes** to avoid confusion.

(2) $$A = \begin{bmatrix} | & & | \\ L(\mathbf{e}_1) & \cdots & L(\mathbf{e}_n) \\ | & & | \end{bmatrix}$$

- **STANDARD BASES OF P_n (REVIEW):**

NOTATION: A standard basis is denoted by $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$, where $\mathbf{e}_j$ is the j^{th} standard basis vector.

With a standard basis, the coefficients in the linear combination are simply the entries of the vector:

$\star$ $$a_0 + a_1x + a_2x^2 + a_3x_3 = a_0 \underbrace{(1)}_{\mathbf{e}_1} + a_1 \underbrace{(x)}_{\mathbf{e}_2} + a_2 \underbrace{(x^2)}_{\mathbf{e}_3} + a_3 \underbrace{(x^3)}_{\mathbf{e}_4} = a_0\mathbf{e}_1 + a_1\mathbf{e}_2 + a_2\mathbf{e}_3 + a_3\mathbf{e}_4$$

- **ISOMORPHISM (REVIEW):**

Vector spaces $\mathbb{R}^4$, $\mathbb{R}^{2\times 2}$, P_3 are all isomorphic to each other since there exists isomorphisms L_1, L_2 as shown below:

Let $L_1 : \mathbb{R}^{2\times 2} \to \mathbb{R}^4$ s.t. $L_1\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$, Let $L_2 : P_3 \to \mathbb{R}^4$ s.t. $L_2\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$

NOTATION: $P_3 \simeq \mathbb{R}^4$ means P_3 is isomorphic to $\mathbb{R}^4$

- **FINDING STANDARD MATRIX FOR CALCULUS (DERIVATIVE/INTEGRAL) OF POLYNOMIALS:**

GIVEN: Linear Transformation $L : P_n \to P_m$ s.t. L involves derivatives and/or integrals.

TASK: Find Standard Matrix A s.t. $L(\mathbf{x}) = A\mathbf{x}$

(1) Compute images $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_{n+1})$.

(2) Produce the isomorphisms of these images in P_m to images in $\mathbb{R}^{m+1}$:

$$[L(\mathbf{e}_1)]_{\mathcal{E}'}, [L(\mathbf{e}_2)]_{\mathcal{E}'}, \ldots, [L(\mathbf{e}_{n+1})]_{\mathcal{E}'}$$

(3) $$A = \begin{bmatrix} | & | & & | \\ [L(\mathbf{e}_1)]_{\mathcal{E}'} & [L(\mathbf{e}_2)]_{\mathcal{E}'} & \cdots & [L(\mathbf{e}_{n+1})]_{\mathcal{E}'} \\ | & | & & | \end{bmatrix}$$

$\mathcal{E} = \{1, t, t^2, \ldots, t^n\} \equiv \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n+1}\}$ is the standard ordered basis for P_n.

$\mathcal{E}' = \{1, t, t^2, \ldots, t^m\} \equiv \{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_{m+1}\}$ is the standard ordered basis for P_m.

IMPORTANT: **Always write polynomials in ascending order:** $3 - t + 5t^2$

EX 6.3.1: Let linear transformations $L_1, L_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$L_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad L_1\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad L_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 6x_2 \\ x_1 + 2x_2 \end{bmatrix}.$$

(a) Find the standard matrices A_1, A_2 such that $L_1(\mathbf{x}) = A_1\mathbf{x}$ and $L_2(\mathbf{x}) = A_2\mathbf{x}$.

(b) Find the compositions $L_1 \circ L_2$ and $L_2 \circ L_1$.

(c) Find the inverses L_1^{-1} and L_2^{-1}, or indicate that they do not exist.

EX 6.3.2: Find std matrix A for linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^4$ s.t. $L\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 14 \\ -2 \\ 6 \end{bmatrix}$, $L\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 16 \\ -5 \\ 7 \\ 12 \end{bmatrix}$

EX 6.3.3: Find standard matrix A for linear transformation $L : P_2 \to P_2$ s.t. $L(p) = t^2 p''(t)$.

EX 6.3.4: Find standard matrix A for linear transformation $L : P_2 \to P_3$ s.t. $L(p) = \int p(t)\ dt$. (Do not include '$+C$')

EX 6.3.5: Find standard matrix A for linear transformation $L : P_2 \to \mathbb{R}$ s.t. $L(p) = \int_0^1 p(t)\ dt$.