EX 7.1.2: Let square matrix $A = \begin{bmatrix} 10 & -8 & -5 \\ 12 & -10 & -6 \\ -2 & 2 & 1 \end{bmatrix}$ with characteristic polynomial $p_A(\lambda) = \lambda(\lambda+1)(\lambda-2)$.

(a) Find the eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$ of A.

$$p_A(\lambda) = 0 \implies \lambda(\lambda+1)(\lambda-2) = 0 \implies \lambda \in \{0, -1, 2\} \implies \boxed{\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 2}$$

(b) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}, E_{\lambda_3}$ of A.

$$E_{\lambda_1} = \text{NulSp}(A - \lambda_1 I) \implies A - \lambda_1 I = A + I = \begin{bmatrix} (10+1) & -8 & -5 \\ 12 & (-10+1) & -6 \\ -2 & 2 & (1+1) \end{bmatrix} = \begin{bmatrix} 11 & -8 & -5 \\ 12 & -9 & -6 \\ -2 & 2 & 2 \end{bmatrix}$$

$$\left[(A-\lambda_1 I) \,\middle|\, \vec{\mathbf{0}}\right] = \left[\begin{array}{ccc|c} 11 & -8 & -5 & 0 \\ 12 & -9 & -6 & 0 \\ -2 & 2 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -2 & 2 & 2 & 0 \\ 12 & -9 & -6 & 0 \\ 11 & -8 & -5 & 0 \end{array}\right] \xrightarrow[(\frac{1}{3})R_2 \to R_2]{(-\frac{1}{2})R_1 \to R_1} \left[\begin{array}{ccc|c} \boxed{1} & -1 & -1 & 0 \\ 4 & -3 & -2 & 0 \\ 11 & -8 & -5 & 0 \end{array}\right]$$

$$\xrightarrow[(-11)R_1 + R_3 \to R_3]{(-4)R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -1 & -1 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right] \xrightarrow{(-3)R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} \boxed{1} & -1 & -1 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Column 3 has no pivot
x_3 is a free variable
Let $x_3 = t$

$$\implies \begin{cases} x_1 = -t \\ x_2 = -2t \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \quad \therefore \boxed{E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right\}}$$

$$E_{\lambda_2} = \text{NulSp}(A - \lambda_2 I) \implies A - \lambda_2 I = A = \begin{bmatrix} 10 & -8 & -5 \\ 12 & -10 & -6 \\ -2 & 2 & 1 \end{bmatrix}$$

$$\left[(A-\lambda_2 I) \,\middle|\, \vec{\mathbf{0}}\right] = \left[\begin{array}{ccc|c} 10 & -8 & -5 & 0 \\ 12 & -10 & -6 & 0 \\ -2 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 12 & -10 & -6 & 0 \\ 10 & -8 & -5 & 0 \end{array}\right] \xrightarrow[(\frac{1}{2})R_2 \to R_2]{(-\frac{1}{2})R_1 \to R_1} \left[\begin{array}{ccc|c} \boxed{1} & -1 & -1/2 & 0 \\ 6 & -5 & -3 & 0 \\ 10 & -8 & -5 & 0 \end{array}\right]$$

$$\xrightarrow[(-10)R_1 + R_3 \to R_3]{(-6)R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -1 & -1/2 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \xrightarrow[R_2 + R_1 \to R_1]{(-2)R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1/2 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\text{RREF}(A - \lambda_2 I) \,\middle|\, \vec{\mathbf{0}}\right]$$

Column 3 has no pivot
x_3 is a free variable
Let $x_3 = 2t$

$$\implies \begin{cases} x_1 = t \\ x_2 = 0 \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \therefore \boxed{E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right\}}$$

$$E_{\lambda_3} = \text{NulSp}(A - \lambda_3 I) \implies A - \lambda_3 I = A - 2I = \begin{bmatrix} (10-2) & -8 & -5 \\ 12 & (-10-2) & -6 \\ -2 & 2 & (1-2) \end{bmatrix} = \begin{bmatrix} 8 & -8 & -5 \\ 12 & -12 & -6 \\ -2 & 2 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 8 & -8 & -5 & 0 \\ 12 & -12 & -6 & 0 \\ -2 & 2 & -1 & 0 \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(Gauss-Jordan steps omitted to conserve space.)

Column 2 has no pivot
x_2 is a free variable
Let $x_2 = t$

$$\implies \begin{cases} x_1 = t \\ x_3 = 0 \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \therefore \boxed{E_{\lambda_3} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}}$$

(c) Find eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ of A.

$$\mathbf{x}_1 = (\text{basis vector of } E_{\lambda_1}) = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = (\text{basis vector of } E_{\lambda_2}) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = (\text{basis vector of } E_{\lambda_3}) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

EX 7.1.3: Let sparse square matrix $A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -2 & 2 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

$$\det(A-\lambda I) = \begin{vmatrix} (-3-\lambda) & 0 & 0 \\ 0 & (3-\lambda) & -1 \\ 0 & -2 & (2-\lambda) \end{vmatrix} = (-3-\lambda)\begin{vmatrix} (3-\lambda) & -1 \\ -2 & (2-\lambda) \end{vmatrix} = (-3-\lambda)\Big[(3-\lambda)(2-\lambda)-2\Big]$$

$$= (-3-\lambda)\Big[6-5\lambda+\lambda^2-2\Big] = (-3-\lambda)(\lambda^2-5\lambda+4) = (-3-\lambda)(\lambda-4)(\lambda-1)$$

$$p_A(\lambda) = \det(\lambda I - A) = (-1)^3\det(A-\lambda I) = (-1)\Big[(-3-\lambda)(\lambda-4)(\lambda-1)\Big] = \boxed{(\lambda+3)(\lambda-4)(\lambda-1)}$$

(b) Find the eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$ of A.

$$p_A(\lambda) = 0 \implies (\lambda+3)(\lambda-4)(\lambda-1) = 0 \implies \lambda \in \{-3, 4, 1\} \implies \boxed{\lambda_1 = -3, \quad \lambda_2 = 1, \quad \lambda_3 = 4}$$

(c) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}, E_{\lambda_3}$ of A.

$$E_{\lambda_1} = \text{NulSp}(A-\lambda_1 I) \implies A - \lambda_1 I = A + 3I = \begin{bmatrix} (-3+3) & 0 & 0 \\ 0 & (3+3) & -1 \\ 0 & -2 & (2+3) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & -1 \\ 0 & -2 & 5 \end{bmatrix}$$

$$\Big[(A-\lambda_1 I) \;\Big|\; \vec{\mathbf{0}}\Big] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 6 & -1 & 0 \\ 0 & -2 & 5 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 0 & -2 & 5 & 0 \\ 0 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{3R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 0 & -2 & 5 & 0 \\ 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{(\frac{1}{14})R_2 \to R_2} \left[\begin{array}{ccc|c} 0 & -2 & 5 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{(-5)R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 0 & -2 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{(-\frac{1}{2})R_1 \to R_1} \left[\begin{array}{ccc|c} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Column 1 has no pivot
x_1 is a free variable
Let $x_1 = t$

$$\implies \begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \therefore \boxed{E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}}$$

$$E_{\lambda_2} = \text{NulSp}(A-\lambda_2 I) \implies A - \lambda_2 I = A - I = \begin{bmatrix} (-3-1) & 0 & 0 \\ 0 & (3-1) & -1 \\ 0 & -2 & (2-1) \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\Big[(A-\lambda_2 I) \;\Big|\; \vec{\mathbf{0}}\Big] = \left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right] \xrightarrow{R_2+R_3 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[(\frac{1}{2})R_2 \to R_2]{(-\frac{1}{4})R_1 \to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Column 3 has no pivot
x_3 is a free variable
Let $x_3 = 2t$

$$\implies \begin{cases} x_1 = 0 \\ x_2 = t \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \therefore \boxed{E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\}}$$

$$E_{\lambda_3} = \text{NulSp}(A-\lambda_3 I) \implies A - \lambda_3 I = A - 4I = \begin{bmatrix} (-3-4) & 0 & 0 \\ 0 & (3-4) & -1 \\ 0 & -2 & (2-4) \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\Big[(A-\lambda_3 I) \;\Big|\; \vec{\mathbf{0}}\Big] = \left[\begin{array}{ccc|c} -7 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right] \xrightarrow{(-2)R_2+R_3 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -7 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[(-1)R_2 \to R_2]{(-\frac{1}{7})R_1 \to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Column 3 has no pivot
x_3 is a free variable
Let $x_3 = t$

$$\implies \begin{cases} x_1 = 0 \\ x_2 = -t \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \therefore \boxed{E_{\lambda_3} = \text{span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}}$$

(d) Find eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ of A.

$\mathbf{x}_1 = (\text{basis vector of } E_{\lambda_1}) = (1,0,0)^T$, $\mathbf{x}_2 = (\text{basis vector of } E_{\lambda_2}) = (0,1,2)^T$, $\mathbf{x}_3 = (0,-1,1)^T$