EX 7.1.5: Let square matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

Since A is diagonal, $(A - \lambda I)$ is diagonal $\implies$ $\det(A - \lambda I) =$ product of its main diagonal entries:

$$\det(A-\lambda I) = \begin{vmatrix} (3-\lambda) & 0 & 0 \\ 0 & (3-\lambda) & 0 \\ 0 & 0 & (4-\lambda) \end{vmatrix} = (3-\lambda)(3-\lambda)(4-\lambda) \implies p_A(\lambda) = (-1)^3 \det(A-\lambda I) = \boxed{(\lambda-3)^2(\lambda-4)}$$

(b) Find the eigenvalues $\lambda_1 < \lambda_2$ of A. What are the algebraic multiplicities of λ_1, λ_2?

$$p_A(\lambda) = 0 \implies (\lambda-3)^2(\lambda-4) = 0 \implies \begin{array}{l} \boxed{\lambda_1 = 3} \implies \text{AM}[\lambda_1] = \text{power of } (\lambda-3) \text{ factor in factored } p_A(\lambda) = \boxed{2} \\ \boxed{\lambda_2 = 4} \implies \text{AM}[\lambda_2] = \text{power of } (\lambda-4) \text{ factor in factored } p_A(\lambda) = \boxed{1} \end{array}$$

(c) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}$ of A. What are the geometric multiplicities of λ_1, λ_2?

$$E_{\lambda_1} = \text{NulSp}(A - \lambda_1 I) \implies A - \lambda_1 I = A - 3I = \begin{bmatrix} (3-3) & 0 & 0 \\ 0 & (3-3) & 0 \\ 0 & 0 & (4-3) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[(A-\lambda_1 I) \,\middle|\, \vec{\mathbf{0}}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\text{RREF}(A-\lambda_1 I) \,\middle|\, \vec{\mathbf{0}}\right]$$

Columns 1 & 2 have no pivot
x_1, x_2 are free variables
Let $x_1 = s, x_2 = t$

$$\implies x_3 = 0 \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \boxed{E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}}$$

$$E_{\lambda_2} = \text{NulSp}(A - \lambda_2 I) \implies A - \lambda_2 I = A - 4I = \begin{bmatrix} (3-4) & 0 & 0 \\ 0 & (3-4) & 0 \\ 0 & 0 & (4-4) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[(A-\lambda_2 I) \,\middle|\, \vec{\mathbf{0}}\right] = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[(-1)R_2 \to R_2]{(-1)R_1 \to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\text{RREF}(A-\lambda_2 I) \,\middle|\, \vec{\mathbf{0}}\right]$$

Column 3 has no pivot
x_3 is a free variable
Let $x_3 = t$

$$\implies \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \therefore \boxed{E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}}$$

$\text{GM}[\lambda_1] = \dim(E_{\lambda_1}) = \boxed{2}$

$\text{GM}[\lambda_2] = \dim(E_{\lambda_2}) = \boxed{1}$

(d) Find eigenvector(s) of A. Is A defective?

$\mathbf{x}_{1,1}$ = 1^{st} basis vector of E_{λ_1} = $\boxed{(1,0,0)^T}$

$\mathbf{x}_{1,2}$ = 2^{nd} basis vector of E_{λ_1} = $\boxed{(0,1,0)^T}$

$\mathbf{x}_2$ = Basis vector of E_{λ_2} = $\boxed{(0,0,1)^T}$

$\boxed{\text{Since AM}[\lambda_1] = \text{GM}[\lambda_1] \text{ and AM}[\lambda_2] = \text{GM}[\lambda_2],\ A \text{ is } \underline{\text{not}} \text{ defective.}}$

EX 7.1.6: Let square matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

Since A is upper triangular, $(A - \lambda I)$ is upper triangular $\implies \det(A - \lambda I) =$ product of its main diagonal entries:

$$\det(A-\lambda I) = \begin{vmatrix} (3-\lambda) & 1 & 0 \\ 0 & (3-\lambda) & 0 \\ 0 & 0 & (4-\lambda) \end{vmatrix} = (3-\lambda)(3-\lambda)(4-\lambda) \implies p_A(\lambda) = (-1)^3 \det(A-\lambda I) = \boxed{(\lambda-3)^2(\lambda-4)}$$

(b) Find the eigenvalues $\lambda_1 < \lambda_2$ of A. What are the algebraic multiplicities of λ_1, λ_2?

$$p_A(\lambda) = 0 \implies (\lambda-3)^2(\lambda-4) = 0 \implies \begin{array}{l} \boxed{\lambda_1 = 3} \implies \text{AM}[\lambda_1] = \text{power of } (\lambda-3) \text{ factor in factored } p_A(\lambda) = \boxed{2} \\ \boxed{\lambda_2 = 4} \implies \text{AM}[\lambda_2] = \text{power of } (\lambda-4) \text{ factor in factored } p_A(\lambda) = \boxed{1} \end{array}$$

(c) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}$ of A. What are the geometric multiplicities of λ_1, λ_2?

$$E_{\lambda_1} = \text{NulSp}(A - \lambda_1 I) \implies A - \lambda_1 I = A - 3I = \begin{bmatrix} (3-3) & 1 & 0 \\ 0 & (3-3) & 0 \\ 0 & 0 & (4-3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[(A-\lambda_1 I) \mid \vec{\mathbf{0}}\right] = \left[\begin{array}{ccc|c} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\text{RREF}(A-\lambda_1 I) \mid \vec{\mathbf{0}}\right]$$

Column 1 has no pivot
x_1 is a free variable
Let $x_1 = t$

$$\implies \begin{array}{l} x_2 = 0 \\ x_3 = 0 \end{array} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \therefore \quad \boxed{E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}}$$

$$E_{\lambda_2} = \text{NulSp}(A - \lambda_2 I) \implies A - \lambda_2 I = A - 4I = \begin{bmatrix} (3-4) & 1 & 0 \\ 0 & (3-4) & 0 \\ 0 & 0 & (4-4) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[(A-\lambda_2 I) \mid \vec{\mathbf{0}}\right] = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[(-1)R_2 \to R_2]{(-1)R_1 \to R_1} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Column 3 has no pivot
x_3 is a free variable
Let $x_3 = t$

$$\implies \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \therefore \quad \boxed{E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}}$$

$\text{GM}[\lambda_1] = \dim(E_{\lambda_1}) = \boxed{1}$

$\text{GM}[\lambda_2] = \dim(E_{\lambda_2}) = \boxed{1}$

(d) Find eigenvector(s) of A. Is A defective?

$$\mathbf{x}_1 = \text{Basis vector of } E_{\lambda_1} = \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \qquad \mathbf{x}_2 = \text{Basis vector of } E_{\lambda_2} = \boxed{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$\boxed{\text{Since AM}[\lambda_1] > \text{GM}[\lambda_1],\ A \text{ \underline{is} defective.}}$

EX 7.1.7: Let square matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

Since A is sparse, $(A-\lambda I)$ is sparse $\implies$ Use a cofactor expansion to find $\det(A-\lambda I)$:

$$\det(A-\lambda I) = \begin{vmatrix} (2-\lambda) & 0 & 0 \\ 0 & (0-\lambda) & -1 \\ 0 & 1 & (0-\lambda) \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = (2-\lambda)(\lambda^2+1)$$

$$\implies p_A(\lambda) = (-1)^3 \det(A-\lambda I) = \boxed{(\lambda^2+1)(\lambda-2)}$$

(b) Find the real eigenvalue λ_1 of A. (The other eigenvalue(s) are complex.)

Observe that (λ^2+1) is an **irreducible quadratic** $\implies \lambda^2+1=0$ yields **complex eigenvalues**, so ignore them!

$\therefore\ p_A(\lambda)=0 \implies (\lambda-2)=0 \implies \boxed{\lambda_1 = 2}$

(c) Find the eigenspace E_{λ_1} of A.

$$E_{\lambda_1} = \text{NulSp}(A-\lambda_1 I) \implies A-\lambda_1 I = A-2I = \begin{bmatrix} (2-2) & 0 & 0 \\ 0 & (0-2) & -1 \\ 0 & 1 & (0-2) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\left[(A-\lambda_1 I) \,\middle|\, \vec{\mathbf{0}}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 0 & \boxed{1} & -2 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{(-\frac{1}{5})R_2 \to R_2} \left[\begin{array}{ccc|c} 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{2R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\text{RREF}(A-\lambda_1 I) \,\middle|\, \vec{\mathbf{0}}\right]$$

Column 1 has no pivot
x_1 is a free variable
Let $x_1 = t$

$$\implies \begin{array}{c} x_2 = 0 \\ x_3 = 0 \end{array} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \therefore \boxed{E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}}$$

(d) Find real eigenvector(s) of A. (The other eigenvector(s) are complex.)

$$\mathbf{x}_1 = \text{Basis vector of } E_{\lambda_1} = \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$