EIGENVALUES, EIGENVECTORS, EIGENSPACES: PART I [LARSON 7.1]

- **EIGENVALUES & EIGENVECTORS OF A SQUARE MATRIX (DEFINITION):**

  Let square matrix $A \in \mathbb{R}^{n\times n}$, non-zero vector $\mathbf{x} \in \mathbb{R}^n$, and scalar $\lambda \in \mathbb{R}$.

  Then λ is an **eigenvalue** of A & $\mathbf{x}$ is a corresponding **eigenvector** of A if

  $$\text{(EIG)} \qquad A\mathbf{x} = \lambda\mathbf{x} \qquad (\text{where } \mathbf{x} \neq \vec{\mathbf{0}})$$

  Moreover, the ordered pair $(\lambda, \mathbf{x})$ is called an **eigenpair** of A.

- **MORE REGARDING EIGENVECTORS:** Let square matrix $A \in \mathbb{R}^{n\times n}$. Then:

  (i) A scalar multiple of an eigenvector is also an eigenvector:

  $$\text{(EIG1)} \qquad (\lambda, \mathbf{x}) \text{ is an eigenpair of } A \implies (\lambda, \alpha\mathbf{x}) \text{ is an eigenpair of } A \qquad (\alpha \neq 0)$$

  (ii) The sum of two eigenvectors with same eigenvalue is also an eigenvector:

  $$\text{(EIG2)} \quad (\lambda, \mathbf{x}_1), (\lambda, \mathbf{x}_2) \text{ are eigenpairs of } A \implies (\lambda, \mathbf{x}_1 + \mathbf{x}_2) \text{ is an eigenpair of } A$$

- **EIGENSPACES OF A SQUARE MATRIX:**

  Let square matrix $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$ be an eigenvalue of A.

  Then the λ-**eigenspace** of A is the following subspace of $\mathbb{R}^n$: $\quad E_\lambda := \{\mathbf{x} \in \mathbb{R}^n : (\lambda, \mathbf{x}) \text{ is an eigenpair of } A\} \cup \{\vec{\mathbf{0}}\}$

  i.e. The λ-eigenspace is the set of all eigenvectors of A with eigenvalue λ together with the zero vector

  (but of course $\vec{\mathbf{0}}$ is not an eigenvector.)

- **CHARACTERISTIC POLYNOMIAL FOR A SQUARE MATRIX:**

  Let square matrix $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$ be an eigenvalue of A.

  Then the **characteristic polynomial** of A is defined to be: $\quad p_A(\lambda) := \det(\lambda I - A) = (-1)^n \det(A - \lambda I)$

  Moreover, $p_A(\lambda)$ is a polynomial in λ of degree n.

  Moreover, the equation $p_A(\lambda) = 0$ is called the **characteristic equation** for A.

- **EIGENVALUES, EIGENVECTORS & THE CHARACTERISTIC POLYNOMIAL:**

  Let square matrix $A \in \mathbb{R}^{n\times n}$, non-zero vector $\mathbf{x} \in \mathbb{R}^n$, scalar $\lambda \in \mathbb{R}$. Then:

  (i) λ is an eigenvalue of $A \iff p_A(\lambda) = 0 \iff \det(A - \lambda I) = 0$

  (ii) $\mathbf{x}$ is an eigenvector of $A \iff (\lambda I - A)\mathbf{x} = \vec{\mathbf{0}} \iff (A - \lambda I)\mathbf{x} = \vec{\mathbf{0}}$

- **CASE I: DISTINCT REAL EIGENVALUES:**

  Let square matrix $A \in \mathbb{R}^{n\times n}$ s.t. all eigenvalues are real & distinct. Then:

  A has n eigenpairs $(\lambda_1, \mathbf{x}_1), (\lambda_2, \mathbf{x}_2), \cdots, (\lambda_n, \mathbf{x}_n)$ s.t. $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.

  i.e. Distinct eigenvalue λ_k has one distinct eigenvector $\mathbf{x}_k$ s.t. $A\mathbf{x}_k = \lambda_k \mathbf{x}_k$.

- **CASE I: DISTINCT REAL EIGENVALUES (PROCEDURE):**

  GIVEN: Square Matrix $A \in \mathbb{R}^{n\times n}$ s.t. all eigenvalues are real & distinct.

  TASK: Find the Eigenvalues λ_k, Eigenvectors $\mathbf{x}_k$, Eigenspaces E_{λ_k} of A.

  (1) Find Characteristic Polynomial $p_A(\lambda) = (-1)^n \det(A - \lambda I)$

  (2) Solve Characteristic Eqn $\det(A - \lambda I) = 0$ to find Eigenvalues $\lambda_1, \ldots, \lambda_n$

  (3) Find the Eigenspace for each Eigenvalue λ_k: $\quad E_{\lambda_k} = \text{NulSp}(A - \lambda_k I)$

  (4) Find an Eigenvector for each Eigenvalue λ_k: $\quad \mathbf{x}_k = (\text{basis vector for } E_{\lambda_k})$

  SANITY CHECKS: $A\mathbf{x}_k = \lambda_k \mathbf{x}_k$, $\dim(E_{\lambda_k}) = 1$, $\mathbf{x}_k$'s are distinct and non-zero

- **EIGENVALUES OF TRIANGULAR & DIAGONAL MATRICES:**

  The eigenvalues of a triangular matrix are the main diagonal entries.

  The eigenvalues of a diagonal matrix are the main diagonal entries.

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EIGENVALUES, EIGENVECTORS, EIGENSPACES: PART II [LARSON 7.1]

- **MULTIPLICITIES OF EIGENVALUES (DEFINITION):**

  Let matrix $A \in \mathbb{R}^{n\times n}$ have (repeated) real eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_p$, where $p < n$

  Moreover, let A have the following factored characteristic polynomial

  $$p_A(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_p)^{m_p} \text{ (where } m_1, \ldots, m_p \in \mathbb{Z}_+)$$

  The **algebraic multiplicity (AM)** of eigenvalue λ_k is m_k.

  The **geometric multiplicity (GM)** of eigenvalue λ_k is $\dim(E_{\lambda_k})$.

  i.e. $\text{AM}[\lambda_k] := m_k = \#$ occurrences of $\lambda_k =$ power of factor $(\lambda - \lambda_k)$ in $p_A(\lambda)$.

  i.e. $\text{GM}[\lambda_k] := \dim(E_{\lambda_k}) = \#$ basis vectors of eigenspace E_{λ_k}.

- **DEFECTIVE MATRICES (DEFINITION):**

  Let square matrix $A \in \mathbb{R}^{n\times n}$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$, where $p \leq n$. Then:

  A is a **defective matrix** if at least one eigenvalue λ_k satisfies $\text{AM}[\lambda_k] > \text{GM}[\lambda_k]$

  i.e. There's fewer linearly indep. eigenvectors for λ_k than $\#$ occurrences of λ_k.

  e.g. Matrix $D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is defective since $\lambda_1 = 1$ and $\text{AM}[\lambda_1] = 3, \text{GM}[\lambda_1] = 2 \implies \text{AM}[\lambda_1] > \text{GM}[\lambda_1]$

  e.g. Matrix $F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is defective since $\lambda_1 = 1$ and $\text{AM}[\lambda_1] = 3, \text{GM}[\lambda_1] = 1 \implies \text{AM}[\lambda_1] > \text{GM}[\lambda_1]$

- **CASE II: REPEATED REAL EIGENVALUES (PROCEDURE):**

  GIVEN: Square Matrix $A \in \mathbb{R}^{n\times n}$ s.t. all eigenvalues are real, some repeated.

  TASK: Find the Eigenvalues λ_k, Eigenvectors $\mathbf{x}_k$, Eigenspaces E_{λ_k} of A.

  (1) Find Characteristic Polynomial $p_A(\lambda) = (-1)^n \det(A - \lambda I)$

  (2) Solve Characteristic Eqn $p_A(\lambda) = 0$ to find Eigenvalues $\lambda_1, \ldots, \lambda_p$ $(p < n)$

  (3) Find the Eigenspace for each Eigenvalue λ_k: $E_{\lambda_k} = \text{NulSp}(A - \lambda_k I)$

  (4) Find an Eigenvector for each λ_k.

  If distinct λ_k: $\mathbf{x}_k =$ (basis vector for E_{λ_k})

  If repeated λ_k: $\mathbf{x}_{k,1} = (1^{st}$ basis vector for $E_{\lambda_k})$, $\mathbf{x}_{k,2} = (2^{nd}$ basis vector for $E_{\lambda_k})$, ...

  IMPORTANT: Repeated eigenvalues do not receive different indices!!

  e.g. If A has eigenvalues $4, 2, 2, 2, -1, -1$, then: $\lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 4$

- **INVERTIBILITY & EIGENVALUES:** Let square matrix $A \in \mathbb{R}^{n\times n}$. Then:

  A is invertible $\iff$ All eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ $(p \leq n)$ are non-zero

  A is not invertible $\iff$ At least one eigenvalue $\lambda_k = 0$

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EIGENVALUES, EIGENVECTORS, EIGENSPACES: PART III [LARSON 7.1]

- **IRREDUCIBLE QUADRATICS (DEFN):** Quadratic ax^2+bx+c is an **irreducible quadratic** $\iff$ $b^2-4ac<0$.
  i.e., the linear factors of an irreducible quadratic are **complex** (not real): (Recall that **imaginary number** $i=\sqrt{-1}$)

  ⋆ x^2+1 is irreducible since $x^2+1=(x-i)(x+i)$ $\quad[b^2-4ac=-4<0]$

  ⋆ x^2-1 is reducible since $x^2-1=(x-1)(x+1)$ $\quad[b^2-4ac=\ \ 4>0]$

  ⋆ x^2+2x+2 is irreducible since $x^2+2x+2=[x+(1-i)][x+(1+i)]$ $\quad[b^2-4ac=-4<0]$

- **FUNDAMENTAL THEOREM OF ALGEBRA (FTA):**
  Every n^{th}-degree **polynomial with complex coefficients** can be factored into n **linear factors with complex coefficients**, some of which may be repeated.

- **COROLLARY TO THE FTA:**
  Every n^{th}-degree **polynomial with real coefficients** can be factored into **linears & irreducible quadratics with real coefficients**.
  What this corollary means for finding eigenvalues is that the characteristic polynomial can always be factored into:

  Linear factors $(\lambda-\lambda_k)$ AND Irreducible quadratics $(\lambda^2+\alpha\lambda+\beta)$.

  e.g. If a 4×4 matrix A has characteristic poly $p_A(\lambda)=(\lambda^2+1)(\lambda-3)(\lambda+4)$,
  then A has real eigenvalues $\lambda_1=-4, \lambda_2=3$ and two complex eigenvalues since λ^2+1 is an irreducible quadratic.

- **CASE III: SOME COMPLEX EIGENVALUES (PROCEDURE):**
  GIVEN: Square Matrix $A\in\mathbb{R}^{n\times n}$ s.t. some eigenvalues are complex.

  TASK: Find the **real** Eigenvalues λ_k, Eigenvectors $\mathbf{x}_k$, Eigenspaces E_{λ_k} of A.

  (1) Find Characteristic Polynomial $p_A(\lambda)=(-1)^n\det(A-\lambda I)$
  (2) Solve Characteristic Eqn $p_A(\lambda)=0$, **ignoring irreducible quadratics**, to find **real** Eigenvalues.
  (3) Find the Eigenspace for each real Eigenvalue λ_k: $E_{\lambda_k}=\text{NulSp}(A-\lambda_k I)$
  (4) Find an Eigenvector for each λ_k.
  If distinct λ_k: $\mathbf{x}_k=$ (basis vector for E_{λ_k})
  If repeated λ_k: $\mathbf{x}_{k,1}=$ (1^{st} basis vector for E_{λ_k}), $\mathbf{x}_{k,2}=$ (2^{nd} basis vector for E_{λ_k}), ...

  IMPORTANT: Repeated eigenvalues do not receive different indices!!

  e.g. If A has eigenvalues $4,2,2,2,-1,-1$, then: $\lambda_1=-1, \lambda_2=2, \lambda_3=4$

CASE IV: ALL COMPLEX EIGENVALUES (PROCEDURE):

The Good News: CASE IV will never be considered in this course!

The Bad News: CASE IV will show up in higher math courses (e.g. Differential Equations II)

Here are some 2×2 matrices that have all complex eigenvalues:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\quad \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix},\quad \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

The standard matrix for linear transformations representing certain rotations will have all complex eigenvalues.
Of course, since all matrices considered will have real entries, a complex eigenvalue will have complex eigenvector(s).

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EX 7.1.1: Let square matrix $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

(b) Find the eigenvalues $\lambda_1 < \lambda_2$ of A.

(c) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}$ of A.

(d) Find eigenvectors $\mathbf{x}_1, \mathbf{x}_2$ of A.

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EX 7.1.2: Let square matrix $A = \begin{bmatrix} 10 & -8 & -5 \\ 12 & -10 & -6 \\ -2 & 2 & 1 \end{bmatrix}$ with characteristic polynomial $p_A(\lambda) = \lambda(\lambda+1)(\lambda-2)$.

(a) Find the eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$ of A.

(b) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}, E_{\lambda_3}$ of A.

(c) Find eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ of A.

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EX 7.1.3: Let sparse square matrix $A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -2 & 2 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

(b) Find the eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$ of A.

(c) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}, E_{\lambda_3}$ of A.

(d) Find eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ of A.

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EX 7.1.4: Let square matrix $A = \begin{bmatrix} 3 & 1/2 \\ -2 & 5 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

(b) Find the eigenvalue λ_1 of A. What is the algebraic multiplicity of λ_1?

(c) Find the eigenspace E_{λ_1} of A. What is the geometric multiplicity of λ_1?

(d) Find eigenvector(s) of A. Is A defective?

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EX 7.1.5: Let square matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

(b) Find the eigenvalues $\lambda_1 < \lambda_2$ of A. What are the algebraic multiplicities of λ_1, λ_2?

(c) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}$ of A. What are the geometric multiplicities of λ_1, λ_2?

(d) Find eigenvector(s) of A. Is A defective?

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EX 7.1.6: Let square matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

(b) Find the eigenvalues $\lambda_1 < \lambda_2$ of A. What are the algebraic multiplicities of λ_1, λ_2?

(c) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}$ of A. What are the geometric multiplicities of λ_1, λ_2?

(d) Find eigenvector(s) of A. Is A defective?

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EX 7.1.7: Let square matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

(b) Find the real eigenvalue λ_1 of A. (The other eigenvalue(s) are complex.)

(c) Find the eigenspace E_{λ_1} of A.

(d) Find real eigenvector(s) of A. (The other eigenvector(s) are complex.)

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