EX 7.2.3: Let matrix $A = \begin{bmatrix} -4 & 6 & 3 \\ -3 & 5 & 3 \\ 0 & 0 & -1 \end{bmatrix}$. Find a 3×3 invertible matrix X and 3×3 diagonal matrix Λ s.t. $A = X\Lambda X^{-1}$.

$$\det(A-\lambda I) = \begin{vmatrix} (-4-\lambda) & 6 & 3 \\ -3 & (5-\lambda) & 3 \\ 0 & 0 & (-1-\lambda) \end{vmatrix} = (-1-\lambda)\begin{vmatrix} (-4-\lambda) & 6 \\ -3 & (5-\lambda) \end{vmatrix} = (-1-\lambda)\Big[(-4-\lambda)(5-\lambda)-(6)(-3)\Big]$$

$$= (-1-\lambda)(\lambda^2-\lambda-2) = (-1-\lambda)(\lambda-2)(\lambda+1)$$

$\therefore\ p_A(\lambda) = \det(\lambda I - A) = (-1)^3\det(A-\lambda I) = (\lambda+1)^2(\lambda-2) \implies$ Eigenvalues of A are $\lambda_1 = -1, \lambda_2 = 2$

$$E_{\lambda_1} = \text{NulSp}(A-\lambda_1 I) \implies A - \lambda_1 I = A + I = \begin{bmatrix} -3 & 6 & 3 \\ -3 & 6 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Big[(A-\lambda_1 I)\ \Big|\ \vec{\mathbf{0}}\Big] = \left[\begin{array}{ccc|c} -3 & 6 & 3 & 0 \\ -3 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{(-1)R_1+R_2\to R_2} \left[\begin{array}{ccc|c} -3 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{(-\frac{1}{3})R_1\to R_1} \left[\begin{array}{ccc|c} \boxed{1} & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Columns 2 & 3 have no pivot

x_2, x_3 are free variables

Let $x_2 = s, x_3 = t$

Then $x_1 - 2x_2 - x_3 = 0$

$\implies x_1 = 2s + t$

$$\implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2s+t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \therefore\ E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$E_{\lambda_2} = \text{NulSp}(A-\lambda_2 I) \implies A - \lambda_2 I = A - 2I = \begin{bmatrix} -6 & 6 & 3 \\ -3 & 3 & 3 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\Big[(A-\lambda_2 I)\ \Big|\ \vec{\mathbf{0}}\Big] = \left[\begin{array}{ccc|c} -6 & 6 & 3 & 0 \\ -3 & 3 & 3 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow[(-\frac{1}{3})R_3\to R_3]{R_1\leftrightarrow R_2} \left[\begin{array}{ccc|c} -3 & 3 & 3 & 0 \\ -6 & 6 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{(-2)R_1+R_2\to R_2} \left[\begin{array}{ccc|c} -3 & 3 & 3 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[(-\frac{1}{3})R_2\to R_2]{(-\frac{1}{3})R_1\to R_1} \left[\begin{array}{ccc|c} \boxed{1} & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{(-1)R_2+R_3\to R_3} \left[\begin{array}{ccc|c} \boxed{1} & -1 & -1 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2+R_1\to R_1} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Column 2 has no pivot

x_2 is a free variable

Let $x_2 = t$

$$\implies \begin{cases} x_1 - x_2 = 0 \\ x_3 = 0 \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \therefore\ E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$\therefore$ Eigenvectors of A are $\mathbf{x}_{1,1} = (2,1,0)^T,\ \mathbf{x}_{1,2} = (1,0,1)^T,\ \mathbf{x}_2 = (1,1,0)^T$

$$\text{Let } X = \begin{bmatrix} | & | & | \\ \mathbf{x}_{1,2} & \mathbf{x}_{1,1} & \mathbf{x}_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \implies \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\implies [X\,|\,I] = \left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 & -1 & -1 \\ 0 & 0 & \boxed{1} & -1 & 2 & 1 \end{array}\right] = [I\,|\,X^{-1}]$$

(Gauss-Jordan steps omitted to save space)

$$\therefore\ A = X\Lambda X^{-1} = \boxed{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix}}$$

NOTE: There's nothing special about this chosen ordering of eigenvector matrix X. Any other ordering of X is fine, provided the same consistent ordering of eigenvalue matrix Λ is used and, of course, the inverse X^{-1} will change. See the 7.2 Slides & Outline for the details about different consistent orderings for X & Λ.