DIAGONALIZATION OF A SQUARE MATRIX [LARSON 7.2]

- **SIMILAR MATRICES (DEFINITION):** Let square matrices $A, B \in \mathbb{R}^{n\times n}$.

  Then A is **similar** to B if $\exists M \in \mathbb{R}^{n\times n}$ s.t. M is invertible and $A = M^{-1}BM$

- **SIMILAR MATRICES HAVE THE SAME EIGENVALUES:** Let square matrices $A, B \in \mathbb{R}^{n\times n}$. Then:

  If A and B are similar, then A and B have the same eigenvalues.

- **DIAGONALIZABLE MATRIX (1^{st} PRINCIPLES DEFINITION):** Let square matrix $A \in \mathbb{R}^{n\times n}$.

  Then A is **diagonalizable** $\Longleftrightarrow$ A is similar to an $n \times n$ diagonal matrix D.

  i.e. A is diagonalizable if $\exists M \in \mathbb{R}^{n\times n}$ s.t. $M^{-1}AM = D$ where D is diagonal.

- **EQUIVALENT CONDITION FOR DIAGONALIZABILITY:** Let square matrix $A \in \mathbb{R}^{n\times n}$. Then:

  A is diagonalizable $\Longleftrightarrow$ A has n linearly independent eigenvectors.

- **SUFFICIENT CONDITION FOR DIAGONALIZABILITY:** Let square matrix $A \in \mathbb{R}^{n\times n}$. Then:

  If A has n distinct eigenvalues, then A is diagonalizable.

- **DIAGONALIZATION OF A SQUARE MATRIX (PROCEDURE):**

  GIVEN: Matrix $A \in \mathbb{R}^{n\times n}$ with all real eigenvalues, some possibly repeated.

  TASK: Diagonalize Square Matrix A, if possible.

  (1) Find the Eigenvalues of A: $\lambda_1, \ldots, \lambda_n$

  (2) Find the Eigenspace E_{λ_k} for each unique Eigenvalue λ_k.
  If AM$[\lambda_k]$ > GM$[\lambda_k]$, then A is not diagonalizable! STOP!!

  (3) Find Eigenvector(s) for each unique Eigenvalue λ_k.

  (4) Let matrix $X \in \mathbb{R}^{n\times n}$ s.t. its columns consist of the eigenvectors.

  (5) Compute X^{-1}: $[X \mid I] \xrightarrow{Gauss-Jordan} [I \mid X^{-1}]$

  (6) Let diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ s.t. the eigenvalues are on its main diagonal.
  The order of eigenvectors in X determine the order of eigenvalues in Λ.

  (7) Form the diagonalization of A: $A = X\Lambda X^{-1}$

  NOTATION: Λ is the capital Greek letter 'lambda'.

- **POWERS OF A SQUARE MATRIX IN DIAGONALIZED FORM:**

  Let square matrix $A \in \mathbb{R}^{n\times n}$ such that A is **diagonalizable**: $A = X\Lambda X^{-1}$

  Moreover, let $k \in \overline{\mathbb{Z}}_+$ be a non-negative integer. Then: $A^k = X\Lambda^k X^{-1}$

- **EIGENVALUES & EIGENVECTORS OF THE INVERSE OF A MATRIX:**

  Let square matrix $A \in \mathbb{R}^{n\times n}$ be invertible. Then:

  $$(\lambda, \mathbf{x}) \text{ is an eigenpair of } A \Longleftrightarrow \left(\frac{1}{\lambda}, \mathbf{x}\right) \text{ is an eigenpair of } A^{-1}$$

CONSISTENCY IS KEY WHEN BUILDING MATRICES X & Λ IN $A = X\Lambda X^{-1}$:

Suppose 3×3 matrix A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$.

Then, A can be diagonalized as $A = X\Lambda X^{-1}$, where:

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

—— OR ——

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_1 & \mathbf{x}_3 & \mathbf{x}_2 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

—— OR ——

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_2 & \mathbf{x}_1 & \mathbf{x}_3 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

—— OR ——

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_1 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

—— OR ——

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_3 & \mathbf{x}_1 & \mathbf{x}_2 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

—— OR ——

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_3 & \mathbf{x}_2 & \mathbf{x}_1 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

Suppose 3×3 matrix A has eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{x}_{1,1}, \mathbf{x}_{1,2}, \mathbf{x}_2$.

Then, A can be diagonalized as $A = X\Lambda X^{-1}$, where:

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_{1,1} & \mathbf{x}_{1,2} & \mathbf{x}_2 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

—— OR ——

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_{1,2} & \mathbf{x}_{1,1} & \mathbf{x}_2 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

—— OR ——

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_{1,1} & \mathbf{x}_2 & \mathbf{x}_{1,2} \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

—— OR ——

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_2 & \mathbf{x}_{1,1} & \mathbf{x}_{1,2} \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

EX 7.2.1: Let matrix $A = \begin{bmatrix} 1 & -3/2 \\ -12 & -2 \end{bmatrix}$. Find a 2×2 invertible matrix X and 2×2 diagonal matrix Λ s.t. $A = X\Lambda X^{-1}$.
If this is not possible, write "A is not diagonalizable."

EX 7.2.2: Let matrix $A = \begin{bmatrix} 18/5 & -1/5 \\ 9/5 & 12/5 \end{bmatrix}$. Find a 2×2 invertible matrix X and 2×2 diagonal matrix Λ s.t. $A = X\Lambda X^{-1}$.

If this is not possible, write "A is not diagonalizable."

EX 7.2.3: Let matrix $A = \begin{bmatrix} -4 & 6 & 3 \\ -3 & 5 & 3 \\ 0 & 0 & -1 \end{bmatrix}$. Find a $3{\times}3$ invertible matrix X and $3{\times}3$ diagonal matrix Λ s.t. $A = X\Lambda X^{-1}$.