EX 7.3.4: Let symmetric matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

$$\det(A-\lambda I) = \begin{vmatrix} (4-\lambda) & 2 \\ 2 & (4-\lambda) \end{vmatrix} = (4-\lambda)^2-(2)(2) = (16-8\lambda+\lambda^2)-4 = \lambda^2-8\lambda+12 \implies \boxed{p_A(\lambda) = \lambda^2 - 8\lambda + 12}$$

(b) Find the eigenvalues $\lambda_1 > \lambda_2$ of A.

$$p_A(\lambda) = 0 \implies \lambda^2 - 8\lambda + 12 = 0 \overset{FACTOR}{\implies} (\lambda-2)(\lambda-6) = 0 \implies \lambda \in \{2, 6\} \implies \boxed{\lambda_1 = 6, \quad \lambda_2 = 2}$$

(c) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}$ of A.

$$E_{\lambda_1} = \text{NulSp}(A-\lambda_1 I) \implies A - \lambda_1 I = A - 6I = \begin{bmatrix} (4-6) & 2 \\ 2 & (4-6) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$\left[(A-\lambda_1 I) \mid \vec{\mathbf{0}}\right] = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{(-\frac{1}{2})R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Column 2 has no pivot
x_2 is a free variable
Let $x_2 = t$

$$\implies \{\ x_1 = t \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \therefore \boxed{E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}}$$

$$E_{\lambda_2} = \text{NulSp}(A-\lambda_2 I) \implies A - \lambda_2 I = A - 2I = \begin{bmatrix} (4-2) & 2 \\ 2 & (4-2) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\left[(A-\lambda_2 I) \mid \vec{\mathbf{0}}\right] = \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{(-1)R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{(\frac{1}{2})R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Column 2 has no pivot
x_2 is a free variable
Let $x_2 = t$

$$\implies \{\ x_1 = -t \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \therefore \boxed{E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}}$$

(d) Find orthogonal eigenvectors $\mathbf{q}_1, \mathbf{q}_2$ of A.

$$\mathbf{q}_1 = (\text{basis vector of } E_{\lambda_1}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \boxed{\mathbf{q}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\mathbf{q}_2 = (\text{basis vector of } E_{\lambda_2}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \boxed{\mathbf{q}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

(e) Find orthonormal eigenvectors $\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2$ of A.

$$\hat{\mathbf{q}}_1 = \frac{\mathbf{q}_1}{||\mathbf{q}_1||_2} = \frac{1}{\sqrt{1^2+1^2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \therefore \boxed{\hat{\mathbf{q}}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}$$

$$\hat{\mathbf{q}}_2 = \frac{\mathbf{q}_2}{||\mathbf{q}_2||_2} = \frac{1}{\sqrt{(-1)^2+1^2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \therefore \boxed{\hat{\mathbf{q}}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}$$

(f) Factor A as $A = Q\Lambda Q^T$ where matrices Q is orthogonal and Λ is diagonal.

$$Q = \begin{bmatrix} | & | \\ \hat{\mathbf{q}}_1 & \hat{\mathbf{q}}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\therefore \boxed{\underbrace{\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{Q^T}}$$

EX 7.3.5: Let symmetric matrix $A = \begin{bmatrix} 2 & 2\sqrt{3} \\ 2\sqrt{3} & 6 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

$$\det(A-\lambda I) = \begin{vmatrix} (2-\lambda) & 2\sqrt{3} \\ 2\sqrt{3} & (6-\lambda) \end{vmatrix} = (2-\lambda)(6-\lambda)-(2\sqrt{3})(2\sqrt{3}) = (12-8\lambda+\lambda^2)-12 = \lambda^2-8\lambda \implies \boxed{p_A(\lambda) = \lambda^2 - 8\lambda}$$

(b) Find the eigenvalues $\lambda_1 > \lambda_2$ of A.

$$p_A(\lambda) = 0 \implies \lambda^2 - 8\lambda = 0 \overset{FACTOR}{\implies} \lambda(\lambda-8) = 0 \implies \lambda \in \{8, 0\} \implies \boxed{\lambda_1 = 8, \quad \lambda_2 = 0}$$

(c) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}$ of A.

$$E_{\lambda_1} = \text{NulSp}(A - \lambda_1 I) \implies A - \lambda_1 I = A - 8I = \begin{bmatrix} (2-8) & 2\sqrt{3} \\ 2\sqrt{3} & (6-8) \end{bmatrix} = \begin{bmatrix} -6 & 2\sqrt{3} \\ 2\sqrt{3} & -2 \end{bmatrix}$$

$$\left[(A-\lambda_1 I) \mid \vec{\mathbf{0}}\right] = \left[\begin{array}{cc|c} -6 & 2\sqrt{3} & 0 \\ 2\sqrt{3} & -2 & 0 \end{array}\right] \xrightarrow{(\frac{1}{\sqrt{3}})R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} -6 & 2\sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{(-\frac{1}{6})R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & -1/\sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Column 2 has no pivot
x_2 is a free variable
Let $x_2 = t\sqrt{3}$

$$\implies \left\{ x_1 = t \right. \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t\sqrt{3} \end{bmatrix} = t\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \quad \therefore \boxed{E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}\right\}}$$

$$E_{\lambda_2} = \text{NulSp}(A - \lambda_2 I) \implies A - \lambda_2 I = A - 0 \cdot I = A = \begin{bmatrix} 2 & 2\sqrt{3} \\ 2\sqrt{3} & 6 \end{bmatrix}$$

$$\left[(A-\lambda_2 I) \mid \vec{\mathbf{0}}\right] = \left[\begin{array}{cc|c} 2 & 2\sqrt{3} & 0 \\ 2\sqrt{3} & 6 & 0 \end{array}\right] \xrightarrow{(-\sqrt{3})R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 2 & 2\sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{(\frac{1}{2})R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & \sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Column 2 has no pivot
x_2 is a free variable
Let $x_2 = t$

$$\implies \left\{ x_1 = -t\sqrt{3} \right. \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t\sqrt{3} \\ t \end{bmatrix} = t\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \quad \therefore \boxed{E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}\right\}}$$

(d) Find orthogonal eigenvectors $\mathbf{q}_1, \mathbf{q}_2$ of A.

$$\mathbf{q}_1 = (\text{basis vector of } E_{\lambda_1}) = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \qquad \boxed{\mathbf{q}_1 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}}$$

$$\mathbf{q}_2 = (\text{basis vector of } E_{\lambda_2}) = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \qquad \boxed{\mathbf{q}_2 = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}}$$

(e) Find orthonormal eigenvectors $\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2$ of A.

$$\hat{\mathbf{q}}_1 = \frac{\mathbf{q}_1}{||\mathbf{q}_1||_2} = \frac{1}{\sqrt{1^2 + (\sqrt{3})^2}}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \qquad \therefore \boxed{\hat{\mathbf{q}}_1 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}}$$

$$\hat{\mathbf{q}}_2 = \frac{\mathbf{q}_2}{||\mathbf{q}_2||_2} = \frac{1}{\sqrt{(-\sqrt{3})^2 + 1^2}}\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix} \qquad \therefore \boxed{\hat{\mathbf{q}}_2 = \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}}$$

(f) Factor A as $A = Q\Lambda Q^T$ where matrices Q is orthogonal and Λ is diagonal.

$$Q = \begin{bmatrix} | & | \\ \hat{\mathbf{q}}_1 & \hat{\mathbf{q}}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \boxed{\underbrace{\begin{bmatrix} 2 & 2\sqrt{3} \\ 2\sqrt{3} & 6 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}}_{Q}\underbrace{\begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix}}_{\Lambda}\underbrace{\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}}_{Q^T}}$$

EX 7.3.6: Let symmetric matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

$$\det(A-\lambda I) = \begin{vmatrix} (2-\lambda) & 2 & 0 \\ 2 & (2-\lambda) & 0 \\ 0 & 0 & (0-\lambda) \end{vmatrix} = (-\lambda)\begin{vmatrix} (2-\lambda) & 2 \\ 2 & (2-\lambda) \end{vmatrix} = (-\lambda)\left[(2-\lambda)^2 - (2)(2)\right] \implies \boxed{p_A(\lambda) = \lambda^2(4-\lambda)}$$

(b) Find the eigenvalues $\lambda_1 > \lambda_2$ of A.

$$p_A(\lambda) = 0 \implies \lambda^2(4-\lambda) = 0 \implies \lambda \in \{4, 0\} \implies \boxed{\lambda_1 = 4, \quad \lambda_2 = 0}$$

(c) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}$ of A.

$$E_{\lambda_1} = \text{NulSp}(A - \lambda_1 I) \implies A - \lambda_1 I = A - 4I = \begin{bmatrix} (2-4) & 2 & 0 \\ 2 & (2-4) & 0 \\ 0 & 0 & (0-4) \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\left[(A-\lambda_1 I) \,\middle|\, \vec{\mathbf{0}}\right] = \left[\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \xrightarrow[(-\frac{1}{4})R_3 \to R_3]{(-\frac{1}{2})R_1 \to R_1} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

Column 2 has no pivot
x_2 is a free variable
Let $x_2 = t$

$$\implies \begin{cases} x_1 = t \\ x_3 = 0 \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \therefore \boxed{E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}}$$

$$E_{\lambda_2} = \text{NulSp}(A - \lambda_2 I) \implies A - \lambda_2 I = A - 0 \cdot I = A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[(A-\lambda_2 I) \,\middle|\, \vec{\mathbf{0}}\right] = \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{(-1)R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{(\frac{1}{2})R_1 \to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Columns 2 & 3 have no pivot
x_2, x_3 are free variables
Let $x_2 = t, \; x_3 = s$
Then $x_1 = -t$

$$\implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \therefore \boxed{E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}}$$

(d) Find orthonormal eigenvectors $\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_{2,1}, \hat{\mathbf{q}}_{2,2}$ of A.

(Normalize each basis vector of $E_{\lambda_1}, E_{\lambda_2}$) $\implies \boxed{\hat{\mathbf{q}}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \hat{\mathbf{q}}_{2,1} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \hat{\mathbf{q}}_{2,2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$

(e) Factor A as $A = Q\Lambda Q^T$ where matrices Q is orthogonal and Λ is diagonal.

$$Q = \begin{bmatrix} | & | & | \\ \hat{\mathbf{q}}_1 & \hat{\mathbf{q}}_{2,1} & \hat{\mathbf{q}}_{2,2} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \boxed{\underbrace{\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q^T}}$$