SYMMETRIC MATRICES & ORTHOGONAL MATRICES [LARSON 7.3]

- **SYMMETRIC MATRIX (DEFINITION):** A square matrix $A \in \mathbb{R}^{n\times n}$ is **symmetric** if $A^T = A$
- **DIAGONAL MATRICES ARE SYMMETRIC:** A diagonal matrix $D \in \mathbb{R}^{n\times n}$ is symmetric.
- **SYMMETRIC MATRIX (PROPERTIES):** Let symmetric matrix $S \in \mathbb{R}^{n\times n}$. Then the following all hold:
  - S is diagonalizable
  - All eigenvalues of S are real
  - If some eigenvalue λ_k repeats, its multiplicities match: $\text{AM}[\lambda_k] = \text{GM}[\lambda_k]$

    i.e. If λ_k occurs j times, then λ_k has j linearly independent eigenvectors: $\mathbf{x}_{k,1},\ \mathbf{x}_{k,2},\ \ldots,\ \mathbf{x}_{k,j-1},\ \mathbf{x}_{k,j}$
- **ORTHOGONAL MATRIX (DEFINITION):**

  A square matrix $Q \in \mathbb{R}^{n\times n}$ is **orthogonal** if Q is invertible and $Q^{-1} = Q^T$
- **DETERMINING IF A SQUARE MATRIX IS ORTHOGONAL:**

  A square matrix $Q \in \mathbb{R}^{n\times n}$ is **orthogonal** $\iff$ $Q^TQ = I$
- **AN ORTHOGONAL MATRIX HAS ORTHONORMAL COLUMNS:**

  A square matrix Q is **orthogonal** $\iff$ its columns form an orthonormal set.
- **ORTHOGONAL PRESERVATION THEOREM:**

  Consider the Euclidean inner product space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ where $\mathbf{v},\mathbf{w},\mathbf{x} \in \mathbb{R}^n$ and

$$\begin{array}{lccc} \text{Inner product} & \langle \mathbf{v},\mathbf{w}\rangle & := & \mathbf{v}^T\mathbf{w} \\ \text{Induced norm} & ||\mathbf{x}|| & := & \sqrt{\langle \mathbf{x},\mathbf{x}\rangle} \\ \text{Induced metric} & d(\mathbf{v},\mathbf{w}) & := & ||\mathbf{v}-\mathbf{w}|| \end{array}$$

  Then orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ preserves inner products, norms & metrics:

$$(i)\ \langle Q\mathbf{v}, Q\mathbf{w}\rangle = \langle \mathbf{v},\mathbf{w}\rangle, \qquad (ii)\ ||Q\mathbf{x}|| = ||\mathbf{x}||, \qquad (iii)\ d(Q\mathbf{v},Q\mathbf{w}) = d(\mathbf{v},\mathbf{w})$$

- **EIGENVECTORS OF A SYMMETRIC MATRIX:**

  Let symmetric matrix $S \in \mathbb{R}^{n\times n}$ have eigenpairs $(\lambda_1,\mathbf{x}_1),(\lambda_2,\mathbf{x}_2)$.

  Then, if eigenvalues λ_1,λ_2 are distinct, then eigenvectors $\mathbf{x}_1,\mathbf{x}_2$ are orthogonal. i.e. If $\lambda_1 \neq \lambda_2$, then $\mathbf{x}_1 \perp \mathbf{x}_2$.

ORTHOGONALLY DIAGONALIZABLE MATRIX (DEFINITION): Let square matrix $A \in \mathbb{R}^{n\times n}$.

Then A is **orthogonally diagonalizable** if $\exists Q \in \mathbb{R}^{n\times n}$ s.t. Q is orthogonal and $Q^TAQ = D$ where D is diagonal.

SYMMETRIC MATRICES ARE ORTHOGONALLY DIAGONALIZABLE:

Square matrix A is orthgonally diagonalizable $\iff$ A is symmetric.

ORTHOGONALLY DIAGONALIZING A SYMMETRIC MATRIX (PROCEDURE):

GIVEN: Symmetric Matrix $S \in \mathbb{R}^{n\times n}$ with some possibly repeated eigenvalues.

TASK: Orthogonally Diagonalize Symmetric Matrix S.

(1) Find the Eigenvalues of S: $\lambda_1,\ldots,\lambda_n$

(2) Find the Eigenspace E_{λ_k} for each unique Eigenvalue λ_k.

(3) Find Unit Eigenvector(s) $\widehat{\mathbf{q}}_k$ for each unique Eigenvalue λ_k: $\widehat{\mathbf{q}}_k = \dfrac{\mathbf{q}_k}{||\mathbf{q}_k||}$

If $\text{AM}[\lambda_k] \geq 2$, then apply Gram-Schmidt on the eigenvectors for λ_k.

(4) Let matrix $Q \in \mathbb{R}^{n\times n}$ s.t. its columns consist of the unit eigenvectors.

(5) Let diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ s.t. the eigenvalues are on its main diagonal.

The order of eigenvectors in Q determine the order of eigenvalues in Λ.

(6) Form the diagonalization of S: $S = Q\Lambda Q^T$

CONSISTENCY IS KEY WHEN BUILDING MATRICES Q & Λ IN $S = Q\Lambda Q^T$:

Let 3×3 symmetric matrix S have eigenvalues $\lambda_1, \lambda_2, \lambda_3$ & eigenvectors $\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2, \widehat{\mathbf{q}}_3$.

Then, S can be orthogonally diagonalized as $S = Q\Lambda Q^T$, where:

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_1 & \widehat{\mathbf{q}}_2 & \widehat{\mathbf{q}}_3 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

—— OR ——

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_2 & \widehat{\mathbf{q}}_1 & \widehat{\mathbf{q}}_3 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

—— OR ——

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_3 & \widehat{\mathbf{q}}_1 & \widehat{\mathbf{q}}_2 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

—— OR ——

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_3 & \widehat{\mathbf{q}}_2 & \widehat{\mathbf{q}}_1 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

Let 3×3 symmetric matrix S have eigenvalues λ_1, λ_2 & eigenvectors $\widehat{\mathbf{q}}_{1,1}, \widehat{\mathbf{q}}_{1,2}, \widehat{\mathbf{q}}_2$.

Then, S can be orthogonally diagonalized as $S = Q\Lambda Q^T$, where:

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_{1,1} & \widehat{\mathbf{q}}_{1,2} & \widehat{\mathbf{q}}_2 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

—— OR ——

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_{1,2} & \widehat{\mathbf{q}}_{1,1} & \widehat{\mathbf{q}}_2 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

—— OR ——

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_{1,1} & \widehat{\mathbf{q}}_2 & \widehat{\mathbf{q}}_{1,2} \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

—— OR ——

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_2 & \widehat{\mathbf{q}}_{1,1} & \widehat{\mathbf{q}}_{1,2} \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

EX 7.3.1: Which of the following 4×4 matrices are symmetric?

$$A = \begin{bmatrix} 1 & 4 & 5 & 5 \\ 4 & 7 & 0 & 9 \\ 5 & 9 & 0 & 1 \\ 5 & 9 & 1 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 5 & 5 \\ 4 & 7 & 0 & 9 \\ 5 & 0 & 0 & 1 \\ 5 & 9 & 1 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & 0 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 9 & 1 & 8 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 7 & 0 \\ 0 & 7 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

EX 7.3.2: Show that the following matrices are orthogonal:

$$Q_1 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$$

EX 7.3.3: Let A be an arbitrary $m \times n$ matrix. Show that $A^T A$ and AA^T are symmetric.

EX 7.3.4: Let symmetric matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

(b) Find the eigenvalues $\lambda_1 > \lambda_2$ of A.

(c) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}$ of A.

(d) Find orthogonal eigenvectors $\mathbf{q}_1, \mathbf{q}_2$ of A.

(e) Find orthonormal eigenvectors $\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2$ of A.

(f) Factor A as $A = Q\Lambda Q^T$ where matrices Q is orthogonal and Λ is diagonal.

EX 7.3.5: Let symmetric matrix $A = \begin{bmatrix} 2 & 2\sqrt{3} \\ 2\sqrt{3} & 6 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

(b) Find the eigenvalues $\lambda_1 > \lambda_2$ of A.

(c) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}$ of A.

(d) Find orthogonal eigenvectors $\mathbf{q}_1, \mathbf{q}_2$ of A.

(e) Find orthonormal eigenvectors $\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2$ of A.

(f) Factor A as $A = Q\Lambda Q^T$ where matrices Q is orthogonal and Λ is diagonal.

EX 7.3.6: Let symmetric matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) Find the characteristic polynomial $p_A(\lambda)$.

(b) Find the eigenvalues $\lambda_1 > \lambda_2$ of A.

(c) Find the eigenspaces $E_{\lambda_1}, E_{\lambda_2}$ of A.

(d) Find orthonormal eigenvectors $\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_{2,1}, \hat{\mathbf{q}}_{2,2}$ of A.

(e) Factor A as $A = Q\Lambda Q^T$ where matrices Q is orthogonal and Λ is diagonal.