EX H.1.1: Let vector $\mathbf{u} := (3,-1)^T \in \mathbb{R}^2$.

(a) Find the same-semiaxis Householder reflector $\hat{\mathbf{h}}_+$ for vector $\mathbf{u}$.

$$||\mathbf{u}||_2 = \sqrt{3^2+(-1)^2} = \sqrt{10}$$

$$\implies \mathbf{h}_+ = \mathbf{u} - \overline{\text{sign}}(u_1)\cdot||\mathbf{u}||_2\cdot\hat{\mathbf{e}}_1 = \begin{bmatrix}3\\-1\end{bmatrix} - \overline{\text{sign}}(3)\cdot\sqrt{10}\cdot\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix} - \sqrt{10}\cdot\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3-\sqrt{10}\\-1\end{bmatrix}$$

$$\implies ||\mathbf{h}_+||_2 = \sqrt{(3-\sqrt{10})^2+(-1)^2} = \sqrt{20-6\sqrt{10}}$$

$$\implies \hat{\mathbf{h}}_+ = \frac{\mathbf{h}_+}{||\mathbf{h}_+||_2} = \boxed{\frac{1}{\sqrt{20-6\sqrt{10}}}\begin{bmatrix}3-\sqrt{10}\\-1\end{bmatrix}}$$

(b) Find the same-semiaxis Householder reflector hyperplane ℓ_+ for vector $\mathbf{u}$.

$$\ell_+ = \left\{\mathbf{x}\in\mathbb{R}^2 : \mathbf{h}_+\cdot\mathbf{x} = 0\right\} = \boxed{\left\{(x_1,x_2)\in\mathbb{R}^2 : (3-\sqrt{10})x_1 - x_2 = 0\right\}}$$

(c) Reflect $\mathbf{u}$ about the same-semiaxis Householder reflector $\hat{\mathbf{h}}_+$ to produce snapped vector $\mathbf{u}_+$.

$$\mathbf{u}_+ = \mathbf{u} - 2\cdot\text{proj}_{\hat{\mathbf{h}}_+}\mathbf{u} = \mathbf{u} - 2(\mathbf{u}\cdot\hat{\mathbf{h}}_+)\hat{\mathbf{h}}_+ = \begin{bmatrix}3\\-1\end{bmatrix} - 2\cdot\left(\frac{(3)(3-\sqrt{10})+(-1)(-1)}{\sqrt{20-6\sqrt{10}}}\right)\cdot\frac{1}{\sqrt{20-6\sqrt{10}}}\begin{bmatrix}3-\sqrt{10}\\-1\end{bmatrix}$$

$$= \begin{bmatrix}3\\-1\end{bmatrix} - 2\cdot\frac{10-3\sqrt{10}}{20-6\sqrt{10}}\begin{bmatrix}3-\sqrt{10}\\-1\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix} - \begin{bmatrix}3-\sqrt{10}\\-1\end{bmatrix} = \boxed{\begin{bmatrix}\sqrt{10}\\0\end{bmatrix}}$$

(d) Find the same-semiaxis Householder reflector matrix H_+ corresponding to $\hat{\mathbf{h}}_+$.

$$\bar{P}_+ = \hat{\mathbf{h}}_+\hat{\mathbf{h}}_+^T = \frac{1}{\sqrt{20-6\sqrt{10}}}\begin{bmatrix}3-\sqrt{10}\\-1\end{bmatrix}\cdot\frac{1}{\sqrt{20-6\sqrt{10}}}\begin{bmatrix}(3-\sqrt{10}) & -1\end{bmatrix}$$

$$= \frac{1}{20-6\sqrt{10}}\begin{bmatrix}(3-\sqrt{10})(3-\sqrt{10}) & (3-\sqrt{10})(-1)\\(-1)(3-\sqrt{10}) & (-1)(-1)\end{bmatrix} = \frac{1}{20-6\sqrt{10}}\begin{bmatrix}19-6\sqrt{10} & -3+\sqrt{10}\\-3+\sqrt{10} & 1\end{bmatrix}$$

$$H_+ = I - 2\bar{P}_+ = \begin{bmatrix}1&0\\0&1\end{bmatrix} - \frac{2}{20-6\sqrt{10}}\begin{bmatrix}19-6\sqrt{10} & -3+\sqrt{10}\\-3+\sqrt{10} & 1\end{bmatrix} = \boxed{\frac{1}{20-6\sqrt{10}}\begin{bmatrix}-18+6\sqrt{10} & 6-2\sqrt{10}\\6-2\sqrt{10} & 18-6\sqrt{10}\end{bmatrix}}$$

(SANITY CHECK) H_+ is symmetric: $H_+^T = H_+$ ✓

(SANITY CHECK) H_+ is orthogonal: $H_+^T H_+ = \frac{1}{760-240\sqrt{10}}\begin{bmatrix}760-240\sqrt{10} & 0\\0 & 760-240\sqrt{10}\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I$ ✓

(CHECK) H_+ reflects $\mathbf{u}$:

$$H_+\mathbf{u} = \frac{1}{20-6\sqrt{10}}\begin{bmatrix}-18+6\sqrt{10} & 6-2\sqrt{10}\\6-2\sqrt{10} & 18-6\sqrt{10}\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix} = \frac{1}{20-6\sqrt{10}}\begin{bmatrix}-60+20\sqrt{10}\\0\end{bmatrix} = \begin{bmatrix}\sqrt{10}\\0\end{bmatrix} = \mathbf{u}_+ \text{ ✓}$$

EX H.1.2: Let vector $\mathbf{u} := (3,-1)^T \in \mathbb{R}^2$.

(a) Find the other-semiaxis Householder reflector $\hat{\mathbf{h}}_-$ for vector $\mathbf{u}$.

$$||\mathbf{u}||_2 = \sqrt{3^2+(-1)^2} = \sqrt{10}$$

$$\Longrightarrow \mathbf{h}_- = \mathbf{u} + \overline{\text{sign}}(u_1)\cdot||\mathbf{u}||_2\cdot\hat{\mathbf{e}}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \overline{\text{sign}}(3)\cdot\sqrt{10}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \sqrt{10}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3+\sqrt{10} \\ -1 \end{bmatrix}$$

$$\Longrightarrow ||\mathbf{h}_-||_2 = \sqrt{(3+\sqrt{10})^2+(-1)^2} = \sqrt{20+6\sqrt{10}}$$

$$\Longrightarrow \hat{\mathbf{h}}_- = \frac{\mathbf{h}_-}{||\mathbf{h}_-||_2} = \boxed{\frac{1}{\sqrt{20+6\sqrt{10}}}\begin{bmatrix} 3+\sqrt{10} \\ -1 \end{bmatrix}}$$

(b) Find the other-semiaxis Householder reflector hyperplane ℓ_- for vector $\mathbf{u}$.

$$\ell_- = \left\{\mathbf{x}\in\mathbb{R}^2 : \mathbf{h}_-\cdot\mathbf{x} = 0\right\} = \boxed{\left\{(x_1,x_2)\in\mathbb{R}^2 : (3+\sqrt{10})x_1 - x_2 = 0\right\}}$$

(c) Reflect $\mathbf{u}$ about the other-semiaxis Householder reflector $\hat{\mathbf{h}}_-$ to produce snapped vector $\mathbf{u}_-$.

$$\mathbf{u}_- = \mathbf{u} - 2\cdot\text{proj}_{\hat{\mathbf{h}}_-}\mathbf{u} = \mathbf{u} - 2(\mathbf{u}\cdot\hat{\mathbf{h}}_-)\hat{\mathbf{h}}_- = \begin{bmatrix} 3 \\ -1 \end{bmatrix} - 2\cdot\left(\frac{(3)(3+\sqrt{10})+(-1)(-1)}{\sqrt{20+6\sqrt{10}}}\right)\cdot\frac{1}{\sqrt{20+6\sqrt{10}}}\begin{bmatrix} 3+\sqrt{10} \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -1 \end{bmatrix} - 2\cdot\frac{10+3\sqrt{10}}{20+6\sqrt{10}}\begin{bmatrix} 3+\sqrt{10} \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 3+\sqrt{10} \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} -\sqrt{10} \\ 0 \end{bmatrix}}$$

(d) Find the other-semiaxis Householder reflector matrix H_- corresponding to $\hat{\mathbf{h}}_-$.

$$\bar{P}_- = \hat{\mathbf{h}}_-\hat{\mathbf{h}}_-^T = \frac{1}{\sqrt{20+6\sqrt{10}}}\begin{bmatrix} 3+\sqrt{10} \\ -1 \end{bmatrix}\cdot\frac{1}{\sqrt{20+6\sqrt{10}}}\begin{bmatrix} (3+\sqrt{10}) & -1 \end{bmatrix}$$

$$= \frac{1}{20+6\sqrt{10}}\begin{bmatrix} (3+\sqrt{10})(3+\sqrt{10}) & (3+\sqrt{10})(-1) \\ (-1)(3+\sqrt{10}) & (-1)(-1) \end{bmatrix} = \frac{1}{20+6\sqrt{10}}\begin{bmatrix} 19+6\sqrt{10} & -3-\sqrt{10} \\ -3-\sqrt{10} & 1 \end{bmatrix}$$

$$H_- = I - 2\bar{P}_- = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{20+6\sqrt{10}}\begin{bmatrix} 19+6\sqrt{10} & -3-\sqrt{10} \\ -3-\sqrt{10} & 1 \end{bmatrix} = \boxed{\frac{1}{20+6\sqrt{10}}\begin{bmatrix} -18-6\sqrt{10} & 6+2\sqrt{10} \\ 6+2\sqrt{10} & 18+6\sqrt{10} \end{bmatrix}}$$

(SANITY CHECK) H_- is symmetric: $H_-^T = H_-$ ✓

(SANITY CHECK) H_- is orthogonal: $H_-^TH_- = \frac{1}{760+240\sqrt{10}}\begin{bmatrix} 760+240\sqrt{10} & 0 \\ 0 & 760+240\sqrt{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ ✓

(CHECK) H_- reflects $\mathbf{u}$: $H_-\mathbf{u} = \frac{1}{20+6\sqrt{10}}\begin{bmatrix} -18-6\sqrt{10} & 6+2\sqrt{10} \\ 6+2\sqrt{10} & 18+6\sqrt{10} \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \frac{1}{20+6\sqrt{10}}\begin{bmatrix} -60-20\sqrt{10} \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{10} \\ 0 \end{bmatrix} = \mathbf{u}_-$ ✓

EX H.1.3: Let vector $\mathbf{v} := (-4, 4, 7)^T \in \mathbb{R}^3$.

(a) Find the same-semiaxis Householder reflector $\hat{\mathbf{h}}_+$ for vector $\mathbf{v}$.

$$||\mathbf{v}||_2 = \sqrt{(-4)^2 + 4^2 + 7^2} = \sqrt{81} = 9$$

$$\implies \mathbf{h}_+ = \mathbf{v} - \overline{\text{sign}}(v_1) \cdot ||\mathbf{v}||_2 \cdot \hat{\mathbf{e}}_1 = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} - \overline{\text{sign}}(-4) \cdot 9 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} + 9 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix}$$

$$\implies ||\mathbf{h}_+||_2 = \sqrt{5^2 + 4^2 + 7^2} = \sqrt{90}$$

$$\implies \hat{\mathbf{h}}_+ = \frac{\mathbf{h}_+}{||\mathbf{h}_+||_2} = \boxed{\frac{1}{\sqrt{90}} \begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix}}$$

(b) Find the same-semiaxis Householder reflector hyperplane ℓ_+ for vector $\mathbf{v}$.

$$\ell_+ = \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{h}_+ \cdot \mathbf{x} = 0 \right\} = \boxed{\left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : 5x_1 + 4x_2 + 7x_3 = 0 \right\}}$$

(c) Reflect $\mathbf{v}$ about the same-semiaxis Householder reflector $\hat{\mathbf{h}}_+$ to produce snapped vector $\mathbf{v}_+$.

$$\mathbf{v}_+ = \mathbf{v} - 2 \cdot \text{proj}_{\hat{\mathbf{h}}_+} \mathbf{v} = \mathbf{v} - 2(\mathbf{v} \cdot \hat{\mathbf{h}}_+)\hat{\mathbf{h}}_+ = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} - 2 \cdot \left(\frac{(-4)(5) + (4)(4) + (7)(7)}{\sqrt{90}} \right) \cdot \frac{1}{\sqrt{90}} \begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} - 2 \cdot \frac{45}{90} \begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix} = \boxed{\begin{bmatrix} -9 \\ 0 \\ 0 \end{bmatrix}}$$

(d) Find the same-semiaxis Householder reflector matrix H_+ corresponding to $\hat{\mathbf{h}}_+$.

$$\bar{P}_+ = \hat{\mathbf{h}}_+ \hat{\mathbf{h}}_+^T = \frac{1}{\sqrt{90}} \begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix} \cdot \frac{1}{\sqrt{90}} [5 \;\; 4 \;\; 7] = \frac{1}{90} \begin{bmatrix} (5)(5) & (5)(4) & (5)(7) \\ (4)(5) & (4)(4) & (4)(7) \\ (7)(5) & (7)(4) & (7)(7) \end{bmatrix} = \frac{1}{90} \begin{bmatrix} 25 & 20 & 35 \\ 20 & 16 & 28 \\ 35 & 28 & 49 \end{bmatrix}$$

$$H_+ = I - 2\bar{P}_+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{90} \begin{bmatrix} 25 & 20 & 35 \\ 20 & 16 & 28 \\ 35 & 28 & 49 \end{bmatrix} = \boxed{\frac{1}{45} \begin{bmatrix} 20 & -20 & -35 \\ -20 & 29 & -28 \\ -35 & -28 & -4 \end{bmatrix}}$$

(SANITY CHECK) H_+ is symmetric: $H_+^T = H_+$ ✓

(SANITY CHECK) H_+ is orthogonal: $H_+^T H_+ = \frac{1}{2025} \begin{bmatrix} 2025 & 0 & 0 \\ 0 & 2025 & 0 \\ 0 & 0 & 2025 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$ ✓

(CHECK) H_+ reflects $\mathbf{v}$: $H_+\mathbf{v} = \frac{1}{45} \begin{bmatrix} 20 & -20 & -35 \\ -20 & 29 & -28 \\ -35 & -28 & -4 \end{bmatrix} \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} = \frac{1}{45} \begin{bmatrix} -405 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ 0 \end{bmatrix} = \mathbf{v}_+$ ✓

EX H.1.4: Let vector $\mathbf{v} := (-4, 4, 7)^T \in \mathbb{R}^3$.

(a) Find the other-semiaxis Householder reflector $\hat{\mathbf{h}}_-$ for vector $\mathbf{v}$.

$||\mathbf{v}||_2 = \sqrt{(-4)^2 + 4^2 + 7^2} = \sqrt{81} = 9$

$$\implies \mathbf{h}_- = \mathbf{v} + \overline{\text{sign}}(v_1) \cdot ||\mathbf{v}||_2 \cdot \hat{\mathbf{e}}_1 = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} + \overline{\text{sign}}(-4) \cdot 9 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} - 9 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -13 \\ 4 \\ 7 \end{bmatrix}$$

$$\implies ||\mathbf{h}_-||_2 = \sqrt{(-13)^2 + 4^2 + 7^2} = \sqrt{234}$$

$$\implies \hat{\mathbf{h}}_- = \frac{\mathbf{h}_-}{||\mathbf{h}_-||_2} = \boxed{\frac{1}{\sqrt{234}} \begin{bmatrix} -13 \\ 4 \\ 7 \end{bmatrix}}$$

(b) Find the other-semiaxis Householder reflector hyperplane ℓ_- for vector $\mathbf{v}$.

$$\ell_- = \left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{h}_- \cdot \mathbf{x} = 0\right\} = \boxed{\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : -13x_1 + 4x_2 + 7x_3 = 0\right\}}$$

(c) Reflect $\mathbf{v}$ about the other-semiaxis Householder reflector $\hat{\mathbf{h}}_-$ to produce snapped vector $\mathbf{v}_-$.

$$\mathbf{v}_- = \mathbf{v} - 2 \cdot \text{proj}_{\hat{\mathbf{h}}_-} \mathbf{v} = \mathbf{v} - 2(\mathbf{v} \cdot \hat{\mathbf{h}}_-)\hat{\mathbf{h}}_- = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} - 2 \cdot \left(\frac{(-4)(-13) + (4)(4) + (7)(7)}{\sqrt{234}}\right) \cdot \frac{1}{\sqrt{234}} \begin{bmatrix} -13 \\ 4 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} - 2 \cdot \frac{117}{234} \begin{bmatrix} -13 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} - \begin{bmatrix} -13 \\ 4 \\ 7 \end{bmatrix} = \boxed{\begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}}$$

(d) Find the other-semiaxis Householder reflector matrix H_- corresponding to $\hat{\mathbf{h}}_-$.

$$\bar{P}_- = \hat{\mathbf{h}}_- \hat{\mathbf{h}}_-^T = \frac{1}{\sqrt{234}} \begin{bmatrix} -13 \\ 4 \\ 7 \end{bmatrix} \cdot \frac{1}{\sqrt{234}} \begin{bmatrix} -13 & 4 & 7 \end{bmatrix} = \frac{1}{234} \begin{bmatrix} (-13)(-13) & (-13)(4) & (-13)(7) \\ (4)(-13) & (4)(4) & (4)(7) \\ (7)(-13) & (7)(4) & (7)(7) \end{bmatrix} = \frac{1}{234} \begin{bmatrix} 169 & -52 & -91 \\ -52 & 16 & 28 \\ -91 & 28 & 49 \end{bmatrix}$$

$$H_- = I - 2\bar{P}_- = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{234} \begin{bmatrix} 169 & -52 & -91 \\ -52 & 16 & 28 \\ -91 & 28 & 49 \end{bmatrix} = \boxed{\frac{1}{117} \begin{bmatrix} -52 & 52 & 91 \\ 52 & 101 & -28 \\ 91 & -28 & 68 \end{bmatrix}}$$

(SANITY CHECK) H_- is symmetric: $H_-^T = H_-$ ✓

(SANITY CHECK) H_- is orthogonal: $$H_-^T H_- = \frac{1}{13689} \begin{bmatrix} 13689 & 0 & 0 \\ 0 & 13689 & 0 \\ 0 & 0 & 13689 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \checkmark$$

(CHECK) H_- reflects $\mathbf{v}$: $$H_- \mathbf{v} = \frac{1}{117} \begin{bmatrix} -52 & 52 & 91 \\ 52 & 101 & -28 \\ 91 & -28 & 68 \end{bmatrix} \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} = \frac{1}{117} \begin{bmatrix} 1053 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} = \mathbf{v}_- \checkmark$$

EX H.1.5: Given the matrix $A = \begin{bmatrix} 8 & -2 \\ 6 & -1 \end{bmatrix} \equiv \begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix}$.

Perform the Full QR Factorization of A by constructing and applying appropriate Householder reflectors.

1^{st}, produce other-semiaxis Householder reflector, H_1, for 1^{st} column of A :

$$||\mathbf{a}_1||_2 = \sqrt{8^2+6^2} = \sqrt{100} = 10 \implies \mathbf{h}_1 = \mathbf{a}_1 + \overline{\text{sign}}(a_{11}) \cdot ||\mathbf{a}_1||_2 \cdot \hat{\mathbf{e}}_1 = \begin{bmatrix} 8 \\ 6 \end{bmatrix} + \overline{\text{sign}}(8) \cdot 10 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$$

$$\implies ||\mathbf{h}_1||_2 = \sqrt{18^2+6^2} = \sqrt{360} \implies \hat{\mathbf{h}}_1 = \frac{\mathbf{h}_1}{||\mathbf{h}_1||_2} = \frac{1}{\sqrt{360}} \cdot \begin{bmatrix} 18 \\ 6 \end{bmatrix}$$

$$\bar{P}_1' = \hat{\mathbf{h}}_1 \hat{\mathbf{h}}_1^T = \frac{1}{360} \cdot \begin{bmatrix} (18)(18) & (18)(6) \\ (6)(18) & (6)(6) \end{bmatrix} = \frac{1}{360} \cdot \begin{bmatrix} 324 & 108 \\ 108 & 36 \end{bmatrix}$$

$$H_1' = I - 2\bar{P}_1' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{360} \cdot \begin{bmatrix} 324 & 108 \\ 108 & 36 \end{bmatrix} = \frac{1}{180} \cdot \begin{bmatrix} -144 & -108 \\ -108 & 144 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = \begin{bmatrix} H_1' \end{bmatrix} = \frac{1}{180} \cdot \begin{bmatrix} -144 & -108 \\ -108 & 144 \end{bmatrix} = \begin{bmatrix} -4/5 & -3/5 \\ -3/5 & 4/5 \end{bmatrix}$$

2^{nd}, apply this Householder reflector, H_1, to A :

$$H_1 A = \frac{1}{180} \cdot \begin{bmatrix} -144 & -108 \\ -108 & 144 \end{bmatrix} \begin{bmatrix} 8 & -2 \\ 6 & -1 \end{bmatrix} = \frac{1}{180} \cdot \begin{bmatrix} -1800 & 396 \\ 0 & 72 \end{bmatrix} = \boxed{\begin{bmatrix} -10 & 11/5 \\ 0 & 2/5 \end{bmatrix} = R}$$

$$\therefore \quad A = QR \quad \text{where} \quad Q = (H_1)^{-1} = H_1^{-1} \overset{ORTH}{=} H_1^T \overset{SYM}{=} H_1 \implies \boxed{Q = \begin{bmatrix} -4/5 & -3/5 \\ -3/5 & 4/5 \end{bmatrix}}$$

$$\therefore \quad A = \begin{bmatrix} 8 & -2 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} -4/5 & -3/5 \\ -3/5 & 4/5 \end{bmatrix} \begin{bmatrix} -10 & 11/5 \\ 0 & 2/5 \end{bmatrix} = QR$$

ALTERNATIVE SOLUTION

1^{st}, produce same-semiaxis Householder reflector, H_1, for 1^{st} column of A :

$$||\mathbf{a}_1||_2 = \sqrt{8^2+6^2} = \sqrt{100} = 10 \implies \mathbf{h}_1 = \mathbf{a}_1 - \overline{\text{sign}}(a_{11}) \cdot ||\mathbf{a}_1||_2 \cdot \hat{\mathbf{e}}_1 = \begin{bmatrix} 8 \\ 6 \end{bmatrix} - \overline{\text{sign}}(8) \cdot 10 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$\implies ||\mathbf{h}_1||_2 = \sqrt{(-2)^2+6^2} = \sqrt{40} \implies \hat{\mathbf{h}}_1 = \frac{\mathbf{h}_1}{||\mathbf{h}_1||_2} = \frac{1}{\sqrt{40}} \cdot \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$\bar{P}_1' = \hat{\mathbf{h}}_1 \hat{\mathbf{h}}_1^T = \frac{1}{40} \cdot \begin{bmatrix} (-2)(-2) & (-2)(6) \\ (6)(-2) & (6)(6) \end{bmatrix} = \frac{1}{40} \cdot \begin{bmatrix} 4 & -12 \\ -12 & 36 \end{bmatrix}$$

$$H_1' = I - 2\bar{P}_1' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{40} \cdot \begin{bmatrix} 4 & -12 \\ -12 & 36 \end{bmatrix} = \frac{1}{20} \cdot \begin{bmatrix} 16 & 12 \\ 12 & -16 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = \begin{bmatrix} H_1' \end{bmatrix} = \frac{1}{20} \cdot \begin{bmatrix} 16 & 12 \\ 12 & -16 \end{bmatrix} = \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}$$

2^{nd}, apply this Householder reflector, H_1, to A :

$$H_1 A = \frac{1}{20} \cdot \begin{bmatrix} 16 & 12 \\ 12 & -16 \end{bmatrix} \begin{bmatrix} 8 & -2 \\ 6 & -1 \end{bmatrix} = \frac{1}{20} \cdot \begin{bmatrix} 200 & -44 \\ 0 & -8 \end{bmatrix} = \boxed{\begin{bmatrix} 10 & -11/5 \\ 0 & -2/5 \end{bmatrix} = R}$$

$$\therefore \quad A = QR \quad \text{where} \quad Q = (H_1)^{-1} = H_1^{-1} \overset{ORTH}{=} H_1^T \overset{SYM}{=} H_1 \implies \boxed{Q = \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}}$$

$$\therefore \quad A = \begin{bmatrix} 8 & -2 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix} \begin{bmatrix} 10 & -11/5 \\ 0 & -2/5 \end{bmatrix} = QR$$

EX H.1.6: Given the matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \equiv \begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix}$.

Perform the Full QR Factorization of A by constructing and applying appropriate Householder reflectors.

1^{st}, produce Householder reflector, H_1, for 1^{st} column of A :

$$||\mathbf{a}_1||_2 = \sqrt{1^2+1^2+1^2} = \sqrt{3} \implies \mathbf{h}_1 = \mathbf{a}_1 + \overline{\text{sign}}(a_{11}) \cdot ||\mathbf{a}_1||_2 \cdot \hat{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \overline{\text{sign}}(1) \cdot \sqrt{3} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{3} \\ 1 \\ 1 \end{bmatrix}$$

$$\implies ||\mathbf{h}_1||_2 = \sqrt{(1+\sqrt{3})^2+1^2+1^2} = \sqrt{6+2\sqrt{3}} \implies \hat{\mathbf{h}}_1 = \frac{\mathbf{h}_1}{||\mathbf{h}_1||_2} = \frac{1}{\sqrt{6+2\sqrt{3}}} \cdot \begin{bmatrix} 1+\sqrt{3} \\ 1 \\ 1 \end{bmatrix}$$

$$\bar{P}_1' = \hat{\mathbf{h}}_1 \hat{\mathbf{h}}_1^T = \frac{1}{6+2\sqrt{3}} \cdot \begin{bmatrix} (1+\sqrt{3})(1+\sqrt{3}) & (1+\sqrt{3})(1) & (1+\sqrt{3})(1) \\ (1)(1+\sqrt{3}) & (1)(1) & (1)(1) \\ (1)(1+\sqrt{3}) & (1)(1) & (1)(1) \end{bmatrix} = \frac{1}{6+2\sqrt{3}} \cdot \begin{bmatrix} (4+2\sqrt{3}) & (1+\sqrt{3}) & (1+\sqrt{3}) \\ (1+\sqrt{3}) & 1 & 1 \\ (1+\sqrt{3}) & 1 & 1 \end{bmatrix}$$

$$H_1' = I - 2\bar{P}_1' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{6+2\sqrt{3}} \cdot \begin{bmatrix} (4+2\sqrt{3}) & (1+\sqrt{3}) & (1+\sqrt{3}) \\ (1+\sqrt{3}) & 1 & 1 \\ (1+\sqrt{3}) & 1 & 1 \end{bmatrix} = \frac{1}{3+\sqrt{3}} \cdot \begin{bmatrix} (-1-\sqrt{3}) & (-1-\sqrt{3}) & (-1-\sqrt{3}) \\ (-1-\sqrt{3}) & (2+\sqrt{3}) & -1 \\ (-1-\sqrt{3}) & -1 & (2+\sqrt{3}) \end{bmatrix}$$

$$H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = \begin{bmatrix} H_1' \end{bmatrix} = \frac{1}{3+\sqrt{3}} \cdot \begin{bmatrix} (-1-\sqrt{3}) & (-1-\sqrt{3}) & (-1-\sqrt{3}) \\ (-1-\sqrt{3}) & (2+\sqrt{3}) & -1 \\ (-1-\sqrt{3}) & -1 & (2+\sqrt{3}) \end{bmatrix}$$

2^{nd}, apply this Householder reflector, H_1, to A :

$$H_1 A = \frac{1}{3+\sqrt{3}} \cdot \begin{bmatrix} (-1-\sqrt{3}) & (-1-\sqrt{3}) & (-1-\sqrt{3}) \\ (-1-\sqrt{3}) & (2+\sqrt{3}) & -1 \\ (-1-\sqrt{3}) & -1 & (2+\sqrt{3}) \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3+\sqrt{3}} \cdot \begin{bmatrix} (-3-3\sqrt{3}) & (2+2\sqrt{3}) \\ 0 & (-1+\sqrt{3}) \\ 0 & (5+3\sqrt{3}) \end{bmatrix} = \begin{bmatrix} -\sqrt{3} & (2/\sqrt{3}) \\ 0 & (-1+2/\sqrt{3}) \\ 0 & (1+2/\sqrt{3}) \end{bmatrix}$$

3^{rd}, produce Householder reflector, H_2, for 1^{st} column of 2x1 lower-right submatrix of $H_1 A$, denoted $A' := \begin{bmatrix} (-1+2/\sqrt{3}) \\ (1+2/\sqrt{3}) \end{bmatrix} \equiv \begin{bmatrix} | \\ \mathbf{a}_1' \\ | \end{bmatrix}$

$$||\mathbf{a}_1'||_2 = \sqrt{\left(-1+\tfrac{2}{\sqrt{3}}\right)^2 + \left(1+\tfrac{2}{\sqrt{3}}\right)^2} = \sqrt{\left(1-\tfrac{4}{\sqrt{3}}+\tfrac{4}{3}\right)+\left(1+\tfrac{4}{\sqrt{3}}+\tfrac{4}{3}\right)} = \sqrt{\tfrac{14}{3}} = \tfrac{\sqrt{42}}{3}$$

$$\implies \mathbf{h}_2 = \mathbf{a}_1' + \overline{\text{sign}}(a_{11}') \cdot ||\mathbf{a}_1'||_2 \cdot \hat{\mathbf{e}}_1 = \begin{bmatrix} \left(-1+\frac{2}{\sqrt{3}}\right) \\ \left(1+\frac{2}{\sqrt{3}}\right) \end{bmatrix} + \overline{\text{sign}}\left(-1+\tfrac{2}{\sqrt{3}}\right) \cdot \sqrt{\tfrac{14}{3}} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \overset{(*)}{=} \begin{bmatrix} \left(-1+\frac{2}{\sqrt{3}}+\frac{\sqrt{14}}{\sqrt{3}}\right) \\ \left(1+\frac{2}{\sqrt{3}}\right) \end{bmatrix}$$

$$(*) \quad -1+\tfrac{2}{\sqrt{3}} = -\tfrac{\sqrt{3}}{\sqrt{3}}+\tfrac{2}{\sqrt{3}} = -\tfrac{\sqrt{3}}{\sqrt{3}}+\tfrac{\sqrt{4}}{\sqrt{3}} = \tfrac{-\sqrt{3}+\sqrt{4}}{\sqrt{3}} > 0 \implies \overline{\text{sign}}\left(-1+\tfrac{2}{\sqrt{3}}\right) = +1$$

$$\implies ||\mathbf{h}_2||_2 = \sqrt{\left(-1+\tfrac{2}{\sqrt{3}}+\tfrac{\sqrt{14}}{\sqrt{3}}\right)^2+\left(1+\tfrac{2}{\sqrt{3}}\right)^2} = \tfrac{\sqrt{28+4\sqrt{14}-2\sqrt{42}}}{\sqrt{3}} \implies \hat{\mathbf{h}}_2 = \frac{\mathbf{h}_2}{||\mathbf{h}_2||_2} = \frac{1}{\sqrt{28+4\sqrt{14}-2\sqrt{42}}} \cdot \begin{bmatrix} 2-\sqrt{3}+\sqrt{14} \\ 2+\sqrt{3} \end{bmatrix}$$

$$\bar{P}_2' = \hat{\mathbf{h}}_2 \hat{\mathbf{h}}_2^T = \frac{1}{28+4\sqrt{14}-2\sqrt{42}} \cdot \begin{bmatrix} (21-4\sqrt{3}+4\sqrt{14}-2\sqrt{42}) & (1+2\sqrt{14}+\sqrt{42}) \\ (1+2\sqrt{14}+\sqrt{42}) & (7+4\sqrt{3}) \end{bmatrix}$$

$$H_2' = I - 2\bar{P}_2' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{28+4\sqrt{14}-2\sqrt{42}} \cdot \begin{bmatrix} (21-4\sqrt{3}+4\sqrt{14}-2\sqrt{42}) & (1+2\sqrt{14}+\sqrt{42}) \\ (1+2\sqrt{14}+\sqrt{42}) & (7+4\sqrt{3}) \end{bmatrix}$$

$$\implies H_2' = \frac{1}{28+4\sqrt{14}-2\sqrt{42}} \cdot \begin{bmatrix} -14+8\sqrt{3}-4\sqrt{14}+2\sqrt{42} & -2-4\sqrt{14}-2\sqrt{42} \\ -2-4\sqrt{14}-2\sqrt{42} & 14-8\sqrt{3}+4\sqrt{14}-2\sqrt{42} \end{bmatrix}$$

$$H_2 = \begin{bmatrix} I_{1\times 1} & \\ & H_2' \end{bmatrix} = \begin{bmatrix} 1 & \\ & H_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-14+8\sqrt{3}-4\sqrt{14}+2\sqrt{42}}{28+4\sqrt{14}-2\sqrt{42}} & \frac{-2-4\sqrt{14}-2\sqrt{42}}{28+4\sqrt{14}-2\sqrt{42}} \\ 0 & \frac{-2-4\sqrt{14}-2\sqrt{42}}{28+4\sqrt{14}-2\sqrt{42}} & \frac{14-8\sqrt{3}+4\sqrt{14}-2\sqrt{42}}{28+4\sqrt{14}-2\sqrt{42}} \end{bmatrix}$$

4^{th}, apply this Householder reflector, H_2, to $H_1 A$:

$$H_2 H_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-14+8\sqrt{3}-4\sqrt{14}+2\sqrt{42}}{28+4\sqrt{14}-2\sqrt{42}} & \frac{-2-4\sqrt{14}-2\sqrt{42}}{28+4\sqrt{14}-2\sqrt{42}} \\ 0 & \frac{-2-4\sqrt{14}-2\sqrt{42}}{28+4\sqrt{14}-2\sqrt{42}} & \frac{14-8\sqrt{3}+4\sqrt{14}-2\sqrt{42}}{28+4\sqrt{14}-2\sqrt{42}} \end{bmatrix} \begin{bmatrix} -\sqrt{3} & (2/\sqrt{3}) \\ 0 & (-1+2/\sqrt{3}) \\ 0 & (1+2/\sqrt{3}) \end{bmatrix} = \boxed{\begin{bmatrix} -\sqrt{3} & 2/\sqrt{3} \\ 0 & -\sqrt{42}/3 \\ 0 & 0 \end{bmatrix} = R}$$

$$\therefore \quad A = QR \quad \text{where} \quad Q = (H_2 H_1)^{-1} = H_1^{-1} H_2^{-1} \overset{ORTH}{=} H_1^T H_2^T \overset{SYM}{=} H_1 H_2$$