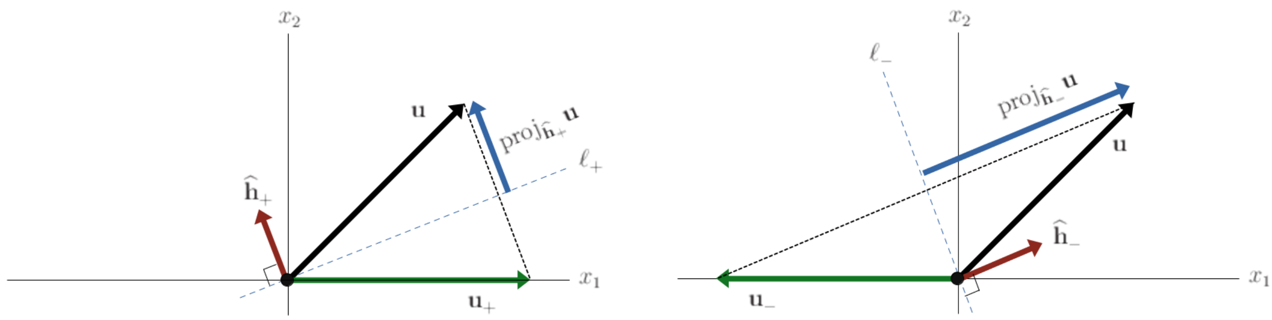
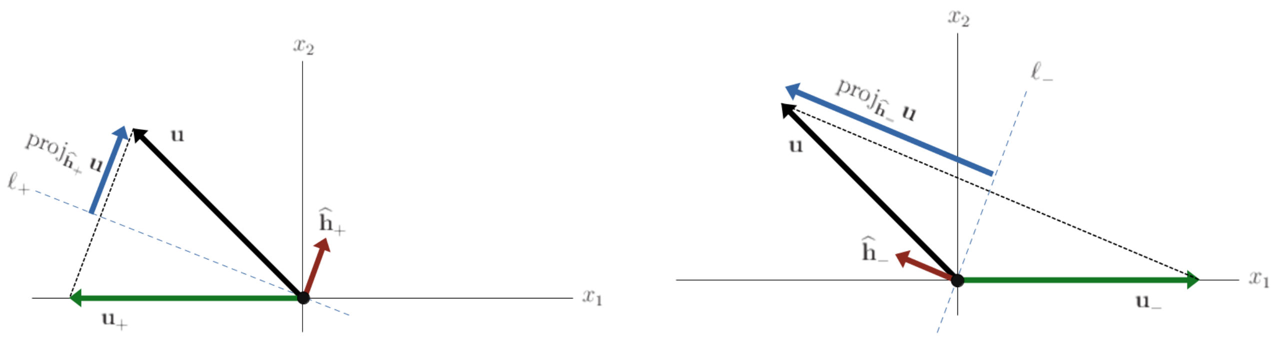
HOUSEHOLDER REFLECTORS OF VECTORS IN $\mathbb{R}^2$

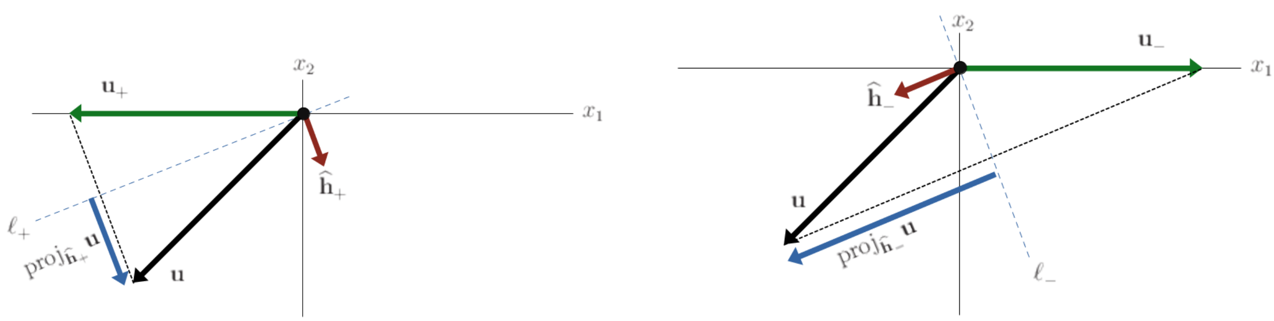
- **HOUSEHOLDER REFLECTORS OF 2D VECTORS IN (QUADRANT I) (VISUALLY):**



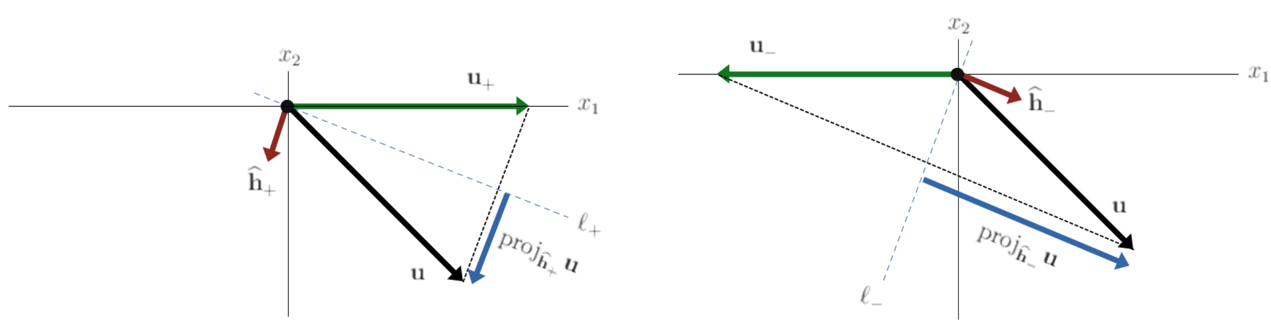
- **HOUSEHOLDER REFLECTORS OF 2D VECTORS IN (QUADRANT II) (VISUALLY):**



- **HOUSEHOLDER REFLECTORS OF 2D VECTORS IN (QUADRANT III) (VISUALLY):**



- **HOUSEHOLDER REFLECTORS OF 2D VECTORS IN (QUADRANT IV) (VISUALLY):**



How to reflect a non-zero vector $\mathbf{u}$ onto the same / other x_1-semiaxis, denoted as $\mathbf{u}_+$ / $\mathbf{u}_-$:

1. Observe that $\mathbf{u}_+/\mathbf{u}_- = \pm||\mathbf{u}||_2 \cdot \hat{\mathbf{e}}_1$ since reflections do not alter magnitude.
2. Form bisecting ray, ℓ_+ / ℓ_-, from origin outward into the same / other x_1-halfplane.
3. Form & normalize vector $\mathbf{h}_+$ / $\mathbf{h}_-$ orthogonal to ℓ_+ / ℓ_- and pointing toward same x_2-halfplane as $\mathbf{u}$.
4. Project vector $\mathbf{u}$ onto unit vector $\hat{\mathbf{h}}_+$ / $\hat{\mathbf{h}}_-$.
5. Subtract twice this projection from vector $\mathbf{u}$, resulting in $\mathbf{u}_+ = \mathbf{u} - 2 \cdot \text{proj}_{\hat{\mathbf{h}}_+}\mathbf{u}$ / $\mathbf{u}_- = \mathbf{u} - 2 \cdot \text{proj}_{\hat{\mathbf{h}}_-}\mathbf{u}$.

HOUSEHOLDER REFLECTORS IN $\mathbb{R}^n$

- **SIGNUM FUNCTION (DEFINITION):**

$$\text{sign}(x) := \begin{cases} +1 & \text{, if } x > 0 \\ 0 & \text{, if } x = 0 \\ -1 & \text{, if } x < 0 \end{cases}$$

- **UPPER-SIGNUM FUNCTION (DEFINITION):**

$$\overline{\text{sign}}(x) := \begin{cases} +1 & \text{, if } x \geq 0 \\ -1 & \text{, if } x < 0 \end{cases}$$

- **LOWER-SIGNUM FUNCTION (DEFINITION):**

$$\underline{\text{sign}}(x) := \begin{cases} +1 & \text{, if } x > 0 \\ -1 & \text{, if } x \leq 0 \end{cases}$$

- **SAME-SEMIAXIS HOUSEHOLDER REFLECTORS IN $\mathbb{R}^n$ (PROCEDURE):** Let vector $\mathbf{u} \in \mathbb{R}^n$. Then:

(1) $\mathbf{h}_+ = \mathbf{u} - \overline{\text{sign}}(u_1) \cdot ||\mathbf{u}||_2 \cdot \hat{\mathbf{e}}_1$
(2) $\hat{\mathbf{h}}_+ = \mathbf{h}_+ / ||\mathbf{h}_+||_2$
(3) $\ell_+ = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{h}_+ \cdot \mathbf{x} = 0\}$

(4) $\bar{P}_+ = \hat{\mathbf{h}}_+ \hat{\mathbf{h}}_+^T$
(5) $H_+ = I - 2\bar{P}_+$

(6) $\mathbf{u}_+ = H_+\mathbf{u} = \mathbf{u} - 2 \cdot \text{proj}_{\hat{\mathbf{h}}_+} \mathbf{u}$

- **OTHER-SEMIAXIS HOUSEHOLDER REFLECTORS IN $\mathbb{R}^n$ (PROCEDURE):** Let vector $\mathbf{u} \in \mathbb{R}^n$. Then:

(1) $\mathbf{h}_- = \mathbf{u} + \overline{\text{sign}}(u_1) \cdot ||\mathbf{u}||_2 \cdot \hat{\mathbf{e}}_1$
(2) $\hat{\mathbf{h}}_- = \mathbf{h}_- / ||\mathbf{h}_-||_2$
(3) $\ell_- = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{h}_- \cdot \mathbf{x} = 0\}$

(4) $\bar{P}_- = \hat{\mathbf{h}}_- \hat{\mathbf{h}}_-^T$
(5) $H_- = I - 2\bar{P}_-$

(6) $\mathbf{u}_- = H_-\mathbf{u} = \mathbf{u} - 2 \cdot \text{proj}_{\hat{\mathbf{h}}_-} \mathbf{u}$

- **HOUSEHOLDER REFLECTORS (PROPERTIES):** Let $H := \hat{\mathbf{h}}\hat{\mathbf{h}}^T \in \mathbb{R}^{n \times n}$ be a Householder Reflector matrix. Then:

(1) H is symmetric: $H^T = H$
(2) H is orthogonal: $H^{-1} = H^T$
(3) H is involutory: $H^2 = I$

(4) $H\hat{\mathbf{h}} = -\hat{\mathbf{h}}$
(5) $H\mathbf{h}_\perp = \mathbf{h}_\perp \ \forall \mathbf{h}_\perp \in \{\hat{\mathbf{h}}\}^\perp$

(6) The eigenvalues of H are: $-1; \underbrace{1, 1, \cdots, 1}_{n-1}$

(7) $\det(H) = -1$

FULL QR FACTORIZATION VIA HOUSEHOLDER REFLECTORS

Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then:

$$\mathbf{a}_1 := (a_{11}, a_{21})^T \implies \mathbf{h}_1 = \mathbf{a}_1 \pm \overline{\text{sign}}(a_{11}) \cdot ||\mathbf{a}_1||_2 \cdot \hat{\mathbf{e}}_1 \implies \hat{\mathbf{h}}_1 = \mathbf{h}_1 / ||\mathbf{h}_1||_2$$

$$\implies H_1' = I - 2\hat{\mathbf{h}}_1\hat{\mathbf{h}}_1^T \implies H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = H_1' \implies H_1 A = \begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}' \end{bmatrix} = R$$

$$\implies Q = (H_1)^{-1} = H_1^{-1} \overset{ORTH}{=} H_1^T \overset{SYM}{=} H_1$$

Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$. Then:

$$\mathbf{a}_1 := (a_{11}, a_{21}, a_{31})^T \implies \mathbf{h}_1 = \mathbf{a}_1 \pm \overline{\text{sign}}(a_{11}) \cdot ||\mathbf{a}_1||_2 \cdot \hat{\mathbf{e}}_1 \implies \hat{\mathbf{h}}_1 = \mathbf{h}_1 / ||\mathbf{h}_1||_2$$

$$\implies H_1' = I - 2\hat{\mathbf{h}}_1\hat{\mathbf{h}}_1^T \implies H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = H_1' \implies H_1 A = \begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}' \\ 0 & a_{32}' \end{bmatrix}$$

$$\mathbf{a}_1' := (a_{22}', a_{32}')^T \implies \mathbf{h}_2 = \mathbf{a}_1' \pm \overline{\text{sign}}(a_{22}') \cdot ||\mathbf{a}_1'||_2 \cdot \hat{\mathbf{e}}_1 \implies \hat{\mathbf{h}}_2 = \mathbf{h}_2 / ||\mathbf{h}_2||_2$$

$$\implies H_2' = I - 2\hat{\mathbf{h}}_2\hat{\mathbf{h}}_2^T \implies H_2 = \begin{bmatrix} I_{1\times 1} & \\ & H_2' \end{bmatrix} = \begin{bmatrix} 1 & \\ & H_2' \end{bmatrix} \implies H_2 H_1 A = \begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}'' \\ 0 & 0 \end{bmatrix} = R$$

$$\implies Q = (H_2 H_1)^{-1} = H_1^{-1} H_2^{-1} \overset{ORTH}{=} H_1^T H_2^T \overset{SYM}{=} H_1 H_2$$

- **<u>WHICH HOUSEHOLDER REFLECTOR??:</u>** Given the Householder Reflector with $\pm$ symbol, which sign to pick:

  $$\mathbf{h}_1 = \mathbf{a}_1^{(1)} \pm \overline{\text{sign}}(a_{11}^{(1)}) \cdot ||\mathbf{a}_1^{(1)}||_2 \cdot \hat{\mathbf{e}}_1$$

  If computing by hand (using exact arithmetic), either $+$ or $-$ is fine, so pick one.

  If computing by computer, always pick the $+$ (i.e. always pick $\mathbf{h}_-$) because the result will be more accurate.

- **<u>FULL QR FACTORIZATION VIA HOUSEHOLDER REFLECTORS (PROCEDURE):</u>**

  <u>GIVEN:</u> Tall or square ($m \geq n$) full column rank matrix $A_{m\times n}$ with columns $\mathbf{a}_k$.

  <u>TASK:</u> Factor $A = QR$ where $Q_{m\times m}$ has orthonormal columns $\widehat{\mathbf{q}}_k$ and $R_{m\times n}$ is upper triangular.

  (0) For each upcoming $\pm$: by computer, always pick $+$; by hand, either $+$ or $-$ is fine, so pick one.

  (1) Build Householder Reflector, H_1, that nullifies sub-diagonal 1^{st} column of A:

  $$\mathbf{a}_1^{(1)} := (a_{11}, \cdots, a_{m1})^T \implies \mathbf{h}_1 = \mathbf{a}_1^{(1)} \pm \overline{\text{sign}}(a_{11}^{(1)}) \cdot ||\mathbf{a}_1^{(1)}||_2 \cdot \hat{\mathbf{e}}_1 \implies \hat{\mathbf{h}}_1 = \mathbf{h}_1 / ||\mathbf{h}_1||_2$$

  $$\implies H_1' = I - 2\hat{\mathbf{h}}_1\hat{\mathbf{h}}_1^T \implies H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = H_1'$$

  (2) For each $j = 2, \cdots, n$:

  Build Householder Reflector, H_j, that nullifies sub-diagonal j^{th} column of $H_j H_{j-1} \cdots H_2 H_1 A := C_j$:

  $$\mathbf{c}_1^{(j)} := \left(c_{jj}^{(j)}, \cdots, c_{mj}^{(j)}\right)^T \implies \mathbf{h}_j = \mathbf{c}_1^{(j)} \pm \overline{\text{sign}}(c_{jj}^{(j)}) \cdot ||\mathbf{c}_1^{(j)}||_2 \cdot \hat{\mathbf{e}}_1 \implies \hat{\mathbf{h}}_j = \mathbf{h}_j / ||\mathbf{h}_j||_2$$

  $$\implies H_j' = I - 2\hat{\mathbf{h}}_j\hat{\mathbf{h}}_j^T \implies H_j = \begin{bmatrix} I_{(j-1)\times(j-1)} & \\ & H_j' \end{bmatrix}$$

  (3) $R = H_n H_{n-1} \cdots H_2 H_1 A$

  (4) $Q = H_1 H_2 \cdots H_{n-1} H_n$

S.J. Leon, *Linear Algebra with Applications*, 9^{th} Ed., Pearson, 2015.

A.S. Householder, "Unitary Triangularization of a Nonsymmetric Matrix", *Journal of the ACM*, **5** (1958), 339-342.

FULL QR FACTORIZATION VIA HOUSEHOLDER REFLECTORS

Let $\quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \implies H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = \begin{bmatrix} H_1' \end{bmatrix}$

$$\implies H_1 A = \begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}' \end{bmatrix} = R$$

$$\implies Q = (H_1)^{-1} = H_1^{-1} \overset{ORTH}{=} H_1^T \overset{SYM}{=} H_1$$

Let $\quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \implies H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = \begin{bmatrix} H_1' \end{bmatrix}$

$$\implies H_1 A = \begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}' \\ 0 & a_{32}' \end{bmatrix} \implies H_2 = \begin{bmatrix} I_{1\times 1} & \\ & H_2' \end{bmatrix} = \begin{bmatrix} 1 & \\ & H_2' \end{bmatrix}$$

$$\implies H_2 H_1 A = \begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}'' \\ 0 & 0 \end{bmatrix} = R$$

$$\implies Q = (H_2 H_1)^{-1} = H_1^{-1} H_2^{-1} \overset{ORTH}{=} H_1^T H_2^T \overset{SYM}{=} H_1 H_2$$

Let $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \implies H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = \begin{bmatrix} H_1' \end{bmatrix}$

$$\implies H_1 A = \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \\ 0 & a_{42}' & a_{43}' \end{bmatrix} \implies H_2 = \begin{bmatrix} I_{1\times 1} & \\ & H_2' \end{bmatrix} = \begin{bmatrix} 1 & \\ & H_2' \end{bmatrix}$$

$$\implies H_2 H_1 A = \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ 0 & a_{22}'' & a_{23}'' \\ 0 & 0 & a_{33}'' \\ 0 & 0 & a_{43}'' \end{bmatrix} \implies H_3 = \begin{bmatrix} I_{2\times 2} & \\ & H_3' \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & H_3' \end{bmatrix}$$

$$\implies H_3 H_2 H_1 A = \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ 0 & a_{22}'' & a_{23}'' \\ 0 & 0 & a_{33}''' \\ 0 & 0 & 0 \end{bmatrix} = R$$

$$\implies Q = (H_3 H_2 H_1)^{-1} = H_1^{-1} H_2^{-1} H_3^{-1} \overset{ORTH}{=} H_1^T H_2^T H_3^T \overset{SYM}{=} H_1 H_2 H_3$$

EX H.1.1: Let vector $\mathbf{u} := (3, -1)^T \in \mathbb{R}^2$.

(a) Find the same-semiaxis Householder reflector $\hat{\mathbf{h}}_+$ for vector $\mathbf{u}$.

(b) Find the same-semiaxis Householder reflector hyperplane ℓ_+ for vector $\mathbf{u}$.

(c) Reflect $\mathbf{u}$ about the same-semiaxis Householder reflector $\hat{\mathbf{h}}_+$ to produce snapped vector $\mathbf{u}_+$.

(d) Find the same-semiaxis Householder reflector matrix H_+ corresponding to $\hat{\mathbf{h}}_+$.

EX H.1.2: Let vector $\mathbf{u} := (3, -1)^T \in \mathbb{R}^2$.

(a) Find the other-semiaxis Householder reflector $\hat{\mathbf{h}}_-$ for vector $\mathbf{u}$.

(b) Find the other-semiaxis Householder reflector hyperplane ℓ_- for vector $\mathbf{u}$.

(c) Reflect $\mathbf{u}$ about the other-semiaxis Householder reflector $\hat{\mathbf{h}}_-$ to produce snapped vector $\mathbf{u}_-$.

(d) Find the other-semiaxis Householder reflector matrix H_- corresponding to $\hat{\mathbf{h}}_-$.

EX H.1.3: Let vector $\mathbf{v} := (-4, 4, 7)^T \in \mathbb{R}^3$.

(a) Find the same-semiaxis Householder reflector $\hat{\mathbf{h}}_+$ for vector $\mathbf{v}$.

(b) Find the same-semiaxis Householder reflector hyperplane ℓ_+ for vector $\mathbf{v}$.

(c) Reflect $\mathbf{v}$ about the same-semiaxis Householder reflector $\hat{\mathbf{h}}_+$ to produce snapped vector $\mathbf{v}_+$.

(d) Find the same-semiaxis Householder reflector matrix H_+ corresponding to $\hat{\mathbf{h}}_+$.

EX H.1.4: Let vector $\mathbf{v} := (-4, 4, 7)^T \in \mathbb{R}^3$.

(a) Find the other-semiaxis Householder reflector $\hat{\mathbf{h}}_-$ for vector $\mathbf{v}$.

(b) Find the other-semiaxis Householder reflector hyperplane ℓ_- for vector $\mathbf{v}$.

(c) Reflect $\mathbf{v}$ about the other-semiaxis Householder reflector $\hat{\mathbf{h}}_-$ to produce snapped vector $\mathbf{v}_-$.

(d) Find the other-semiaxis Householder reflector matrix H_- corresponding to $\hat{\mathbf{h}}_-$.

EX H.1.5: Given the matrix $A = \begin{bmatrix} 8 & -2 \\ 6 & -1 \end{bmatrix} \equiv \begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix}$.

Perform the Full QR Factorization of A by constructing and applying appropriate Householder reflectors.

EX H.1.6: Given the matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \equiv \begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix}$.

Perform the Full QR Factorization of A by constructing and applying appropriate Householder reflectors.