EX S.1.1: Let matrix $A := \begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix}$.

(a) Perform the Full SVD on matrix A.

1^{st}, find the **right-singular vectors** $\hat{\mathbf{v}}_k$ of A by computing the eigenpairs of A^TA in descending eigenvalue order.

NOTE: Since A^TA is symmetric, its eigenvectors, denoted $\mathbf{v}_k$, will be orthogonal, but remember to normalize them to $\hat{\mathbf{v}}_k$.

$$A^TA = \begin{bmatrix} 2 & 0 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\Longrightarrow \text{ Eigenvalues of } A^TA \text{ are (work omitted, see EX 7.3.4): } \mu_1 = 6,\ \mu_2 = 2$$

$$\Longrightarrow \text{ Eigenvectors of } A^TA \text{ are (work omitted): } \hat{\mathbf{v}}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},\ \hat{\mathbf{v}}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

2^{nd}, find the **singular values** of A by taking the square root of the eigenvalues of A^TA :

$$\sigma_1 := \sqrt{\mu_1} = \sqrt{6}, \qquad \sigma_2 := \sqrt{\mu_2} = \sqrt{2}$$

3^{rd}, find the **left-singular vectors** $\hat{\mathbf{u}}_k$ of A by dividing each product $A\hat{\mathbf{v}}_k$ by its singular value σ_k :

$$\hat{\mathbf{u}}_1 := \frac{A\hat{\mathbf{v}}_1}{\sigma_1} = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 3/\sqrt{2} \\ \sqrt{3}/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

$$\hat{\mathbf{u}}_2 := \frac{A\hat{\mathbf{v}}_2}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1/\sqrt{2} \\ \sqrt{3}/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

4^{th}, form the SVD factor matrices U, Σ, V^T :

$$U = \begin{bmatrix} | & | \\ \hat{\mathbf{u}}_1 & \hat{\mathbf{u}}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix},\ \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{2} \end{bmatrix},\ V^T = \begin{bmatrix} \text{---} & \hat{\mathbf{v}}_1 & \text{---} \\ \text{---} & \hat{\mathbf{v}}_2 & \text{---} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore \boxed{\underbrace{\begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{2} \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{V^T}}$$

(b) Using the Full SVD, determine the rank of A.

$r \equiv \text{rank}(A) = (\#\text{ non-zero singular values of } A) = (\#\text{ non-zero main-diagonal entries in } \Sigma) = \boxed{2}$

(c) Using the Full SVD, determine the four fundamental matrix subspaces of A.

$$\text{ColSp}(A) = \text{Span}\{\hat{\mathbf{u}}_1, \cdots, \hat{\mathbf{u}}_r\} = \text{Span}\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2\} \Longrightarrow \boxed{\text{ColSp}(A) = \text{Span}\left\{\begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}\right\}}$$

$$\text{NulSp}(A) = \text{Span}\{\hat{\mathbf{v}}_{r+1}, \cdots, \hat{\mathbf{v}}_n\} = \vec{\mathbf{0}} \text{ (b/c } r = n = 2) \Longrightarrow \boxed{\text{NulSp}(A) = \text{Span}\{\vec{\mathbf{0}}\}}$$

$$\text{ColSp}(A^T) = \text{Span}\{\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_r\} = \text{Span}\{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2\} \Longrightarrow \boxed{\text{ColSp}(A^T) = \text{Span}\left\{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right\}}$$

$$\text{NulSp}(A^T) = \text{Span}\{\hat{\mathbf{u}}_{r+1}, \cdots, \hat{\mathbf{u}}_m\} = \vec{\mathbf{0}} \text{ (b/c } r = m = 2) \Longrightarrow \boxed{\text{NulSp}(A^T) = \text{Span}\{\vec{\mathbf{0}}\}}$$

EX S.1.2: Let matrix $A := \begin{bmatrix} 1 & \sqrt{3} \\ 1 & \sqrt{3} \end{bmatrix}$.

(a) Perform the Full SVD on matrix A.

1^{st}, find the **right-singular vectors** $\hat{\mathbf{v}}_k$ of A by computing the eigenpairs of A^TA in descending eigenvalue order.

NOTE: Since A^TA is symmetric, its eigenvectors, denoted $\mathbf{v}_k$, will be orthogonal, but remember to normalize them to $\hat{\mathbf{v}}_k$.

$$A^TA = \begin{bmatrix} 1 & 1 \\ \sqrt{3} & \sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & \sqrt{3} \\ 1 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 2 & 2\sqrt{3} \\ 2\sqrt{3} & 6 \end{bmatrix}$$

$$\implies \text{Eigenvalues of } A^TA \text{ are (work omitted, see EX 7.3.5): } \mu_1 = 8,\ \mu_2 = 0$$

$$\implies \text{Eigenvectors of } A^TA \text{ are (work omitted): } \hat{\mathbf{v}}_1 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix},\ \hat{\mathbf{v}}_2 = \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

2^{nd}, find the **singular values** of A by taking the square root of the eigenvalues of A^TA :

$$\sigma_1 := \sqrt{\mu_1} = \sqrt{8}, \qquad \sigma_2 := \sqrt{\mu_2} = 0$$

3^{rd}, find the **left-singular vectors** $\hat{\mathbf{u}}_k$ of A by dividing each product $A\hat{\mathbf{v}}_k$ by its positive singular value σ_k.

For each zero singular value, find the corresponding left-singular vector by performing Gram-Schmidt orthonormalization.

$$\hat{\mathbf{u}}_1 := \frac{A\hat{\mathbf{v}}_1}{\sigma_1} = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & \sqrt{3} \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Since $\sigma_2 = 0$, perform Gram-Schmidt on standard basis vector $\hat{\mathbf{e}}_2 := (0,1)^T$ with respect to $\hat{\mathbf{u}}_1$:

$$\mathbf{u}_2 \overset{GS}{=} \hat{\mathbf{e}}_2 - \text{proj}_{\hat{\mathbf{u}}_1}\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_2 - \frac{\langle \hat{\mathbf{e}}_2, \hat{\mathbf{u}}_1\rangle_2}{\langle \hat{\mathbf{u}}_1, \hat{\mathbf{u}}_1\rangle_2}\hat{\mathbf{u}}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} \implies \hat{\mathbf{u}}_2 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||_2} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

4^{th}, form the SVD factor matrices U, Σ, V^T :

$$U = \begin{bmatrix} | & | \\ \hat{\mathbf{u}}_1 & \hat{\mathbf{u}}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},\ \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{8} & 0 \\ 0 & 0 \end{bmatrix},\ V^T = \begin{bmatrix} \text{---} & \hat{\mathbf{v}}_1 & \text{---} \\ \text{---} & \hat{\mathbf{v}}_2 & \text{---} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore \boxed{\underbrace{\begin{bmatrix} 1 & \sqrt{3} \\ 1 & \sqrt{3} \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} \sqrt{8} & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}}_{V^T}}$$

(b) Using the Full SVD, determine the rank of A.

$r \equiv \text{rank}(A) = (\#\text{ non-zero singular values of } A) = (\#\text{ non-zero main-diagonal entries in } \Sigma) = \boxed{1}$

(c) Using the Full SVD, determine the four fundamental matrix subspaces of A.

$$\begin{aligned}
\text{ColSp}(A) &= \text{Span}\{\hat{\mathbf{u}}_1, \cdots, \hat{\mathbf{u}}_r\} = \text{Span}\{\hat{\mathbf{u}}_1\} \implies \boxed{\text{ColSp}(A) = \text{Span}\left\{(1/\sqrt{2},\ 1/\sqrt{2})^T\right\}} \\
\text{NulSp}(A) &= \text{Span}\{\hat{\mathbf{v}}_{r+1}, \cdots, \hat{\mathbf{v}}_n\} = \text{Span}\{\hat{\mathbf{v}}_2\} \implies \boxed{\text{NulSp}(A) = \text{Span}\left\{(-\sqrt{3}/2,\ 1/2)^T\right\}} \\
\text{ColSp}(A^T) &= \text{Span}\{\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_r\} = \text{Span}\{\hat{\mathbf{v}}_1\} \implies \boxed{\text{ColSp}(A^T) = \text{Span}\left\{(1/2,\ \sqrt{3}/2)^T\right\}} \\
\text{NulSp}(A^T) &= \text{Span}\{\hat{\mathbf{u}}_{r+1}, \cdots, \hat{\mathbf{u}}_m\} = \text{Span}\{\hat{\mathbf{u}}_2\} \implies \boxed{\text{NulSp}(A^T) = \text{Span}\left\{(-1/\sqrt{2},\ 1/\sqrt{2})^T\right\}}
\end{aligned}$$

EX S.1.3: Let matrix $A := \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix}$.

(a) Perform the Full SVD on matrix A.

1^{st}, find the **right-singular vectors** $\hat{\mathbf{v}}_k$ of A by computing the eigenpairs of A^TA in descending eigenvalue order.

NOTE: Since A^TA is symmetric, its eigenvectors, denoted $\mathbf{v}_k$, will be orthogonal, but remember to normalize them to $\hat{\mathbf{v}}_k$.

$$A^TA = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 8 \\ 8 & 9 \end{bmatrix}$$

$$\implies \text{Eigenvalues of } A^TA \text{ are (work omitted): } \mu_1 = 17,\ \mu_2 = 1$$

$$\implies \text{Eigenvectors of } A^TA \text{ are (work omitted): } \hat{\mathbf{v}}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},\ \hat{\mathbf{v}}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

2^{nd}, find the **singular values** of A by taking the square root of the eigenvalues of A^TA :

$$\sigma_1 := \sqrt{\mu_1} = \sqrt{17}, \qquad \sigma_2 := \sqrt{\mu_2} = \sqrt{1} = 1$$

3^{rd}, find the **left-singular vectors** $\hat{\mathbf{u}}_k$ of A by dividing each product $A\hat{\mathbf{v}}_k$ by its singular value σ_k :

$$\hat{\mathbf{u}}_1 := \frac{A\hat{\mathbf{v}}_1}{\sigma_1} = \frac{1}{\sqrt{17}} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{17}} \begin{bmatrix} 3/\sqrt{2} \\ 3/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3/\sqrt{34} \\ 3/\sqrt{34} \\ 4/\sqrt{34} \end{bmatrix}$$

$$\hat{\mathbf{u}}_2 := \frac{A\hat{\mathbf{v}}_2}{\sigma_2} = \frac{1}{1} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

There are only two singular values, so apply Gram-Schmidt to std basis vector $\hat{\mathbf{e}}_3 := (0,0,1)^T$ w.r.t $\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2$ to get $\hat{\mathbf{u}}_3$:

$$\mathbf{u}_3 \stackrel{GS}{=} \hat{\mathbf{e}}_3 - \text{proj}_{\hat{\mathbf{u}}_1}\hat{\mathbf{e}}_3 - \text{proj}_{\hat{\mathbf{u}}_2}\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_3 - \frac{\langle \hat{\mathbf{e}}_3, \hat{\mathbf{u}}_1 \rangle_2}{\langle \hat{\mathbf{u}}_1, \hat{\mathbf{u}}_1 \rangle_2}\hat{\mathbf{u}}_1 - \frac{\langle \hat{\mathbf{e}}_3, \hat{\mathbf{u}}_2 \rangle_2}{\langle \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_2 \rangle_2}\hat{\mathbf{u}}_2 = \cdots = \begin{bmatrix} -12/\sqrt{34} \\ -12/\sqrt{34} \\ 18/\sqrt{34} \end{bmatrix} \implies \hat{\mathbf{u}}_3 = \begin{bmatrix} -2/\sqrt{17} \\ -2/\sqrt{17} \\ 3/\sqrt{17} \end{bmatrix}$$

4^{th}, form the SVD factor matrices U, Σ, V^T and be sure to pad Σ with a third row of zeros:

$$U = \begin{bmatrix} | & | & | \\ \hat{\mathbf{u}}_1 & \hat{\mathbf{u}}_2 & \hat{\mathbf{u}}_3 \\ | & | & | \end{bmatrix},\quad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix},\quad V^T = \begin{bmatrix} — & \hat{\mathbf{v}}_1 & — \\ — & \hat{\mathbf{v}}_2 & — \end{bmatrix}$$

$$\therefore \boxed{\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 3/\sqrt{34} & 1/\sqrt{2} & -2/\sqrt{17} \\ 3/\sqrt{34} & -1/\sqrt{2} & -2/\sqrt{17} \\ 4/\sqrt{34} & 0 & 3/\sqrt{17} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{V^T}}$$

(b) Using the Full SVD, determine the column space of A and the null space of A^T.

$$\text{ColSp}(A) = \text{Span}\{\hat{\mathbf{u}}_1, \cdots, \hat{\mathbf{u}}_r\} = \text{Span}\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2\} \implies \boxed{\text{ColSp}(A) = \text{Span}\left\{ \begin{bmatrix} 3/\sqrt{34} \\ 3/\sqrt{34} \\ 4/\sqrt{34} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1\sqrt{2} \\ 0 \end{bmatrix} \right\}}$$

$$\text{NulSp}(A^T) = \text{Span}\{\hat{\mathbf{u}}_{r+1}\} = \text{Span}\{\hat{\mathbf{u}}_3\} \implies \boxed{\text{NulSp}(A^T) = (-2/\sqrt{17}, -2/\sqrt{17}, 3/\sqrt{17})^T}$$

EX S.1.4: Let matrix $A := \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix}$. Perform the Full SVD on matrix A.

1^{st}, find the **right-singular vectors** $\hat{\mathbf{v}}_k$ of A by computing the eigenpairs of A^TA in descending eigenvalue order.

NOTE: Since A^TA is symmetric, its eigenvectors, denoted $\mathbf{v}_k$, will be orthogonal, but remember to normalize them to $\hat{\mathbf{v}}_k$.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 & 3\sqrt{2} \\ 3\sqrt{2} & 6 \end{bmatrix}$$

$$\implies \text{Eigenvalues of } A^TA \text{ are (work omitted): } \mu_1 = 9,\ \mu_2 = 0$$

$$\implies \text{Eigenvectors of } A^TA \text{ are (work omitted): } \hat{\mathbf{v}}_1 = \begin{bmatrix} \sqrt{1/3} \\ \sqrt{2/3} \end{bmatrix},\ \hat{\mathbf{v}}_2 = \begin{bmatrix} -\sqrt{2/3} \\ \sqrt{1/3} \end{bmatrix}$$

2^{nd}, find the **singular values** of A by taking the square root of the eigenvalues of A^TA :

$$\sigma_1 := \sqrt{\mu_1} = \sqrt{9} = 3, \qquad \sigma_2 := \sqrt{\mu_2} = \sqrt{0} = 0$$

3^{rd}, find the **left-singular vectors** $\hat{\mathbf{u}}_k$ of A by dividing each product $A\hat{\mathbf{v}}_k$ by its singular value σ_k :

$$\hat{\mathbf{u}}_1 := \frac{A\hat{\mathbf{v}}_1}{\sigma_1} = \frac{1}{3}\begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{1/3} \\ \sqrt{2/3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} \sqrt{3} \\ \sqrt{3} \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Since $\sigma_2 = 0$, perform Gram-Schmidt on standard basis vector $\hat{\mathbf{e}}_2 := (0,1,0)^T$ with respect to $\hat{\mathbf{u}}_1$:

$$\mathbf{u}_2 \overset{GS}{=} \hat{\mathbf{e}}_2 - \text{proj}_{\hat{\mathbf{u}}_1}\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_2 - \frac{\langle \hat{\mathbf{e}}_2, \hat{\mathbf{u}}_1 \rangle_2}{\langle \hat{\mathbf{u}}_1, \hat{\mathbf{u}}_1 \rangle_2}\hat{\mathbf{u}}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{3}}\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \end{bmatrix} \implies \hat{\mathbf{u}}_2 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||_2} = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

There are only two singular values, so apply Gram-Schmidt to std basis vector $\hat{\mathbf{e}}_3 := (0,0,1)^T$ w.r.t $\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2$ to get $\hat{\mathbf{u}}_3$:

$$\mathbf{u}_3 \overset{GS}{=} \hat{\mathbf{e}}_3 - \text{proj}_{\hat{\mathbf{u}}_1}\hat{\mathbf{e}}_3 - \text{proj}_{\hat{\mathbf{u}}_2}\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_3 - \frac{\langle \hat{\mathbf{e}}_3, \hat{\mathbf{u}}_1 \rangle_2}{\langle \hat{\mathbf{u}}_1, \hat{\mathbf{u}}_1 \rangle_2}\hat{\mathbf{u}}_1 - \frac{\langle \hat{\mathbf{e}}_3, \hat{\mathbf{u}}_2 \rangle_2}{\langle \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_2 \rangle_2}\hat{\mathbf{u}}_2 = \cdots = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} \implies \hat{\mathbf{u}}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

4^{th}, form the SVD factor matrices U, Σ, V^T and be sure to pad Σ with a third row of zeros:

$$U = \begin{bmatrix} | & | & | \\ \hat{\mathbf{u}}_1 & \hat{\mathbf{u}}_2 & \hat{\mathbf{u}}_3 \\ | & | & | \end{bmatrix},\ \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix},\ V^T = \begin{bmatrix} \text{---} & \hat{\mathbf{v}}_1 & \text{---} \\ \text{---} & \hat{\mathbf{v}}_2 & \text{---} \end{bmatrix}$$

$$\therefore \underbrace{\begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \sqrt{1/3} & \sqrt{2/3} \\ -\sqrt{2/3} & \sqrt{1/3} \end{bmatrix}}_{V^T}$$