SINGULAR VALUE DECOMPOSITION (SVD)

- **FULL SINGULAR VALUE DECOMPOSITION (MOTIVATION):**

  Recall that only symmetric matrices have an ortho-eigen-decomposition: $S_{n\times n} = Q\Lambda Q^T$

  Question: If this decomposition is relaxed by allowing two unalike orthogonal matrices but forcing...

  ...all the middle's main diagonal entries to be real #'s can such a decomposition exist for all rectangular matrices???

  i.e. Is this possible: $A_{m\times n} = U\Sigma V^T$ s.t. $U^TU = I_{m\times m}$, $V^TV = I_{n\times n}$, $\Sigma_{m\times n}$ is real-entried diagonal

  Answer: Turns out, YES! Let's derive it!

- **FULL SINGULAR VALUE DECOMPOSITION (PROPERTIES OF A^TA & AA^T):**

  Let A be a tall or square $m\times n$ matrix. Then:

  (i) A^TA and AA^T are both symmetric.

  (ii) The eigenvalues of A^TA and AA^T are all real.

  (iii) The eigenvectors of A^TA and AA^T are orthonormal.

  (iv) $(\mu, \mathbf{v}) \neq (0, \vec{\mathbf{0}})$ is an eigenpair of $A^TA \iff (\mu, A\mathbf{v})$ is an eigenpair of AA^T.

  (v) The eigenvalues of A^TA and AA^T are non-negative.

  (vi) $\text{NulSp}(A^TA) = \text{NulSp}(A)$, $\text{NulSp}(AA^T) = \text{NulSp}(A^T)$

  (vii) $\text{ColSp}(A^TA) = \text{ColSp}(A^T)$, $\text{ColSp}(AA^T) = \text{ColSp}(A)$

- **FULL SINGULAR VALUE DECOMPOSITION (DERIVATION):**

  Given: $m\times n$ rectangular matrix A such that $\text{rank}(A) = r$.

$$\text{Factor } A^TA = VMV^T \text{ s.t. } \left\{\begin{array}{ccc} V^TV = VV^T = I_{n\times n} & , & M = \text{diag}(\mu_1, \cdots, \mu_n) \\ \langle \hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j\rangle_2 = \delta_{ij} & , & A^TA\hat{\mathbf{v}}_k = \mu_k\hat{\mathbf{v}}_k \ \forall k \leq n \end{array}\right\} \text{ Then:}$$

$$||A\hat{\mathbf{v}}_k||_2^2 = (A\hat{\mathbf{v}}_k)^T(A\hat{\mathbf{v}}_k) \overset{T}{=} \hat{\mathbf{v}}_k^TA^TA\hat{\mathbf{v}}_k \overset{EIG}{=} \hat{\mathbf{v}}_k^T\mu_k\hat{\mathbf{v}}_k = \mu_k(\hat{\mathbf{v}}_k^T\hat{\mathbf{v}}_k) = \mu_k||\hat{\mathbf{v}}_k||_2^2 = \mu_k\cdot 1 = \mu_k$$

$$||A\hat{\mathbf{v}}_k||_2 = \sqrt{\mu_k} := \sigma_k \implies \hat{\mathbf{u}}_k := A\hat{\mathbf{v}}_k/\sigma_k. \text{ Descend-sort-label the } \textbf{singular values} \text{ of } A \text{ like so: } \sigma_1 \geq \cdots \geq \sigma_r > 0$$

$$\langle \hat{\mathbf{u}}_i, \hat{\mathbf{u}}_j\rangle_2 = \hat{\mathbf{u}}_i^T\hat{\mathbf{u}}_j = \left(\frac{A\hat{\mathbf{v}}_i}{\sigma_i}\right)^T\left(\frac{A\hat{\mathbf{v}}_j}{\sigma_j}\right) \overset{T}{=} \frac{\hat{\mathbf{v}}_i^TA^TA\hat{\mathbf{v}}_j}{\sigma_i\sigma_j} \overset{EIG}{=} \frac{\mu_j}{\sigma_i\sigma_j}\cdot\hat{\mathbf{v}}_i^T\hat{\mathbf{v}}_j \overset{\perp}{=} \delta_{ij} \implies \langle \hat{\mathbf{u}}_i, \hat{\mathbf{u}}_j\rangle_2 = \delta_{ij}$$

$$\therefore \left\{\begin{array}{cccl} A\hat{\mathbf{v}}_k & = & \sigma_k\hat{\mathbf{u}}_k & \text{for } k \leq r \\ A\hat{\mathbf{v}}_k & = & \vec{\mathbf{0}} & \text{for } r < k \leq n \end{array}\right. \implies AV = U\Sigma \implies \boxed{A = U\Sigma V^T}$$

$$\text{Factor } AA^T = UMU^T \text{ s.t. } \left\{\begin{array}{ccc} U^TU = UU^T = I_{m\times m} & , & M = \text{diag}(\mu_1, \cdots, \mu_m) \\ \langle \hat{\mathbf{u}}_i, \hat{\mathbf{u}}_j\rangle_2 = \delta_{ij} & , & AA^T\hat{\mathbf{u}}_k = \mu_k\hat{\mathbf{u}}_k \ \forall k \leq m \end{array}\right\} \text{ Then:}$$

$$||A^T\hat{\mathbf{u}}_k||_2^2 = (A^T\hat{\mathbf{u}}_k)^T(A^T\hat{\mathbf{u}}_k) \overset{T}{=} \hat{\mathbf{u}}_k^T(AA^T)\hat{\mathbf{u}}_k \overset{EIG}{=} \hat{\mathbf{u}}_k^T\mu_k\hat{\mathbf{u}}_k = \mu_k(\hat{\mathbf{u}}_k^T\hat{\mathbf{u}}_k) = \mu_k||\hat{\mathbf{u}}_k||_2^2 = \mu_k\cdot 1 = \mu_k$$

$$||A^T\hat{\mathbf{u}}_k||_2 = \sqrt{\mu_k} := \sigma_k \implies \hat{\mathbf{v}}_k := A^T\hat{\mathbf{u}}_k/\sigma_k. \text{ Descend-sort-label the } \textbf{singular values} \text{ of } A \text{ like so: } \sigma_1 \geq \cdots \geq \sigma_r > 0$$

$$\langle \hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j\rangle_2 = \hat{\mathbf{v}}_i^T\hat{\mathbf{v}}_j = \left(\frac{A^T\hat{\mathbf{u}}_i}{\sigma_i}\right)^T\left(\frac{A^T\hat{\mathbf{u}}_j}{\sigma_j}\right) \overset{T}{=} \frac{\hat{\mathbf{u}}_i^TAA^T\hat{\mathbf{u}}_j}{\sigma_i\sigma_j} \overset{EIG}{=} \frac{\mu_j}{\sigma_i\sigma_j}\cdot\hat{\mathbf{u}}_i^T\hat{\mathbf{u}}_j \overset{\perp}{=} \delta_{ij} \implies \langle \hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j\rangle_2 = \delta_{ij}$$

$$\therefore \left\{\begin{array}{cccl} A^T\hat{\mathbf{u}}_k & = & \sigma_k\hat{\mathbf{v}}_k & \text{for } 1 \leq k \leq r \\ A^T\hat{\mathbf{u}}_k & = & \vec{\mathbf{0}} & \text{for } r < k \leq m \end{array}\right. \implies A^TU = V\Sigma^T \implies A^T = V\Sigma^TU^T \overset{T}{\implies} \boxed{A = U\Sigma V^T}$$

$$A_{m\times n} = \underbrace{\begin{bmatrix} | & & | \\ \hat{\mathbf{u}}_1 & \cdots & \hat{\mathbf{u}}_m \\ | & & | \end{bmatrix}}_{m\times m} \underbrace{\left[\begin{array}{c:c} \dot{\Sigma}_{r\times r} & \\ \hdashline & O_{(m-r)\times(n-r)} \end{array}\right]}_{m\times n} \underbrace{\begin{bmatrix} — & \hat{\mathbf{v}}_1 & — \\ & \vdots & \\ — & \hat{\mathbf{v}}_n & — \end{bmatrix}}_{n\times n}$$

$$\text{where: } U^{-1} = U^T, \ V^{-1} = V^T, \ \dot{\Sigma} := \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \text{ with } \sigma_1 \geq \cdots \geq \sigma_r > 0$$

SINGULAR VALUE DECOMPOSITION (SVD)

- **FULL SINGULAR VALUE DECOMPOSITION (PROCEDURE):**

GIVEN: $m \times n$ tall ($m \geq n$) or square ($m = n$) matrix A with column rank $r \leq n$:

TASK: Factor $A = U\Sigma V^T$ where: $U^TU = I_{m\times m}$, $V^TV = I_{n\times n}$, Σ is $m \times n$ diagonal

1. Find the n eigenvalues $\mu_1 \geq \cdots \geq \mu_r > \mu_{r+1} = \cdots = \mu_n = 0$ of symmetric matrix A^TA.
2. Find the r **right-singular vectors** of A: $\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_r$ = eigenvectors of A^TA w.r.t. $\mu_1, \cdots, \mu_r$
3. Find the r **singular values** of A: $\sigma_1 := \sqrt{\mu_1},\ \cdots,\ \sigma_r := \sqrt{\mu_r}$
4. Find the r **left-singular vectors** of A: $\hat{\mathbf{u}}_1 := A\hat{\mathbf{v}}_1/\sigma_1,\ \cdots,\ \hat{\mathbf{u}}_r := A\hat{\mathbf{v}}_r/\sigma_r$
5. If $r < n$, then use Gram-Schmidt on std basis vectors $\hat{\mathbf{e}}_{r+1}, \cdots, \hat{\mathbf{e}}_n \in \mathbb{R}^n$ to find: $\hat{\mathbf{v}}_{r+1}, \cdots, \hat{\mathbf{v}}_n$
6. If $r < m$, then use Gram-Schmidt on std basis vectors $\hat{\mathbf{e}}_{r+1}, \cdots, \hat{\mathbf{e}}_m \in \mathbb{R}^m$ to find: $\hat{\mathbf{u}}_{r+1}, \cdots, \hat{\mathbf{u}}_m$
7. Form the matrices comprising the Full SVD like so:

$$A_{m\times n} = \underbrace{\begin{bmatrix} | & & | \\ \hat{\mathbf{u}}_1 & \cdots & \hat{\mathbf{u}}_m \\ | & & | \end{bmatrix}}_{m\times m} \underbrace{\left[\begin{array}{c:c} \dot{\Sigma}_{r\times r} & \\ \hdashline & O_{(m-r)\times(n-r)} \end{array}\right]}_{m\times n} \underbrace{\begin{bmatrix} \text{---} & \hat{\mathbf{v}}_1 & \text{---} \\ & \vdots & \\ \text{---} & \hat{\mathbf{v}}_n & \text{---} \end{bmatrix}}_{n\times n} \equiv U\Sigma V^T$$

where: $O_{(m-r)\times(n-r)}$ is zero matrix and $\dot{\Sigma} := \text{diag}(\sigma_1, \cdots, \sigma_r)$ with $\sigma_1 \geq \cdots \geq \sigma_r > 0$

- **THE FOUR FUNDAMENTAL MATRIX SUBSPACES VIA FULL SVD:**

GIVEN: $m \times n$ tall/square ($m \geq n$) matrix A with column rank $r \leq n$ and Full SVD $A = U\Sigma V^T$.

Then, the four fundamental matrix subspaces of A are (directly via the Full SVD):

- ⋆ $\text{ColSp}(A) = \text{Span}\{\hat{\mathbf{u}}_1, \cdots, \hat{\mathbf{u}}_r\}$ $\quad$ $\text{NulSp}(A) = \text{Span}\{\hat{\mathbf{v}}_{r+1}, \cdots, \hat{\mathbf{v}}_n\}$
- ⋆ $\text{ColSp}(A^T) = \text{Span}\{\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_r\}$ $\quad$ $\text{NulSp}(A^T) = \text{Span}\{\hat{\mathbf{u}}_{r+1}, \cdots, \hat{\mathbf{u}}_m\}$

Moreover, the relations among these fundamental subspaces are now immediate:

- ⋆ $\dim \text{ColSp}(A) = \dim \text{ColSp}(A^T) = r$
- ⋆ $\mathbb{R}^m = \text{ColSp}(A) \oplus \text{NulSp}(A^T)$
- ⋆ $\mathbb{R}^n = \text{ColSp}(A^T) \oplus \text{NulSp}(A)$

- **REDUCED SINGULAR VALUE DECOMPOSITION (PROCEDURE):**

GIVEN: $m \times n$ tall ($m \geq n$) or square ($m = n$) matrix A with column rank $r \leq n$:

TASK: Factor $A = \hat{U}\dot{\Sigma}\hat{V}^T$ where: $\hat{U}^T\hat{U} = I_{m\times m}$, $\hat{V}^T\hat{V} = I_{n\times n}$, $\dot{\Sigma}$ is $r \times r$ diagonal

1. Find the n eigenvalues $\mu_1 \geq \cdots \geq \mu_r > \mu_{r+1} = \cdots = \mu_n = 0$ of symmetric matrix A^TA.
2. Find the r **right-singular vectors** of A: $\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_r$ = eigenvectors of A^TA w.r.t. $\mu_1, \cdots, \mu_r$
3. Find the r **singular values** of A: $\sigma_1 := \sqrt{\mu_1},\ \cdots,\ \sigma_r := \sqrt{\mu_r}$
4. Find the r **left-singular vectors** of A: $\hat{\mathbf{u}}_1 := A\hat{\mathbf{v}}_1/\sigma_1,\ \cdots,\ \hat{\mathbf{u}}_r := A\hat{\mathbf{v}}_r/\sigma_r$
5. Form the matrices comprising the Reduced SVD like so:

$$A_{m\times n} = \underbrace{\begin{bmatrix} | & & | \\ \hat{\mathbf{u}}_1 & \cdots & \hat{\mathbf{u}}_r \\ | & & | \end{bmatrix}}_{m\times r} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}}_{r\times r} \underbrace{\begin{bmatrix} \text{---} & \hat{\mathbf{v}}_1 & \text{---} \\ & \vdots & \\ \text{---} & \hat{\mathbf{v}}_r & \text{---} \end{bmatrix}}_{r\times n} \equiv \hat{U}\dot{\Sigma}\hat{V}^T$$

where: $\sigma_1 \geq \cdots \geq \sigma_r > 0$

EX S.1.1: Let matrix $A := \begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix}$.

(a) Perform the Full SVD on matrix A.

(b) Using the Full SVD, determine the rank of A.

(c) Using the Full SVD, determine the four fundamental matrix subspaces of A.

EX S.1.2: Let matrix $A := \begin{bmatrix} 1 & \sqrt{3} \\ 1 & \sqrt{3} \end{bmatrix}$.

(a) Perform the Full SVD on matrix A.

(b) Using the Full SVD, determine the rank of A.

(c) Using the Full SVD, determine the four fundamental matrix subspaces of A.

EX S.1.3: Let matrix $A := \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix}$.

(a) Perform the Full SVD on matrix A.

(c) Using the Full SVD, determine the column space of A and the null space of A^T.

EX S.1.4: Let matrix $A := \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix}$. Perform the Full SVD on matrix A.