Linear Systems ($A\mathbf{x} = \mathbf{b}$): Intro, Interpretation

Linear Algebra

Josh Engwer

TTU

24 August 2015

What is Elementary Linear Algebra??

Definition

(Linear Algebra)

Elementary Linear Algebra is the study of:

Linear Systems	(Chapters 1-3)
Vector Spaces	(Chapters 4-5)
Linear Transformations	(Chapters 6-7)

All these areas of Linear Algebra involve the use of:

- Scalars
- Vectors
- Matrices

Definition

(Scalar)
A **scalar** is a **real #**: $0, 1, -5, 9/7, 2.4, \sqrt[3]{7}, 6 - \sqrt{2}, \log 7, \pi, e, \sin(\pi/7), \ldots$

Complex numbers will <u>never</u> be considered: $3i$, $5 - 2i$ where $i := \sqrt{-1}$

Linear Systems (Definition)

Definition

(Linear System)

A $m \times n$ **linear system in** $x_1, x_2, \ldots, x_n$ has the following form:

$$\left\{ \begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \right.$$

where scalars $a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, \ldots, a_{mn} \in \mathbb{R}$
and the RHS scalars $b_1, \ldots, b_m \in \mathbb{R}$. [RHS $\equiv$ "**R**ight-**H**and **S**ide"]

The **unknown variables** are $x_1, \ldots, x_n$, **and appear only to the <u>first power</u>**.

i.e., a $m \times n$ linear system has m linear equations & n unknown variables.

Linear systems with one eqn $(m = 1)$ or one unknown $(n = 1)$ are too trivial.

For the purposes of this course, $1 < m, n < 6$.

Linear Systems (Examples)

2×2 linear system in x, y : $\begin{cases} x - y = 5 - \sqrt{2} \\ 2x + y = \sqrt[3]{9} \end{cases}$

2×3 linear system in x, y, z : $\begin{cases} x + z = \pi \\ 3x - y - \frac{1}{2}z = 0 \end{cases}$

3×2 linear system in x_1, x_2 : $\begin{cases} x_1 + x_2 = 1 \\ \left(\sqrt{3}\right) x_1 - x_2 = \frac{3}{2} \\ x_1 = 0 \end{cases}$

3×3 linear system in x_1, x_2, x_3 : $\begin{cases} x_1 + x_2 + x_3 = -2 \\ 2x_1 - x_2 = 10 \\ x_2 - 3x_3 = \pi\sqrt{3} \end{cases}$

Linear Systems (Non-Intuitive Examples)

Consider the following system: $\begin{cases} (\cos\theta)x + (\sin\theta)y = 1 \\ (-\sin\theta)x + (\cos\theta)y = 0 \end{cases}$

This is a 2×2 linear system in x, y (treating θ as **constant**)

Consider the following system: $\begin{cases} k_1x_1 + k_2x_2 + x_3 = 7 \\ k_1x_1 - x_2 + k_3x_3 = -3 \end{cases}$

This is a 2×3 linear system in x_1, x_2, x_3 (treating k_1, k_2, k_3 as **constants**)

Non-Linear Systems & Changes of Variables (CV's)

Definition

(Nonlinear Systems)

A system of equations which is not linear is called a **nonlinear system**.

Occasionally, it's possible to convert a nonlinear system into a linear system by an appropriate **change of variables (CV)**:

CV: Let $u = 1/x$ and $v = \sqrt{y}$. Then:

$$\left\{\begin{array}{rcrcr} 4/x & - & 3\sqrt{y} & = & 1 \\ -1/x & + & \pi\sqrt{y} & = & 0 \end{array}\right. \overset{CV}{\Longleftrightarrow} \left\{\begin{array}{rcrcr} 4u & - & 3v & = & 1 \\ -u & + & \pi v & = & 0 \end{array}\right.$$

Usually, however, it's simply not possible: $\left\{\begin{array}{rcr} x^2y^3 & = & 7 \\ e^x \arctan y & = & -5 \end{array}\right.$

Linear Systems (More Definitions)

Definition

(Underdetermined & Overdetermined Linear Systems)

A $m \times n$ linear system is...

..underdetermined	$\iff$	there's more unknowns than equations $(m < n)$
..overdetermined	$\iff$	there's more equations than unknowns $(m > n)$
..square	$\iff$	there's as many equations as unknowns $(m = n)$

Underdetermined linear system:
$$\begin{cases} x + z = \pi \\ 3x - y - \frac{1}{2}z = 0 \end{cases}$$

Overdetermined linear system:
$$\begin{cases} x_1 + x_2 = 1 \\ \left(\sqrt{3}\right)x_1 - x_2 = \frac{3}{2} \\ x_1 = 0 \end{cases}$$

Square linear system:
$$\begin{cases} x - y = 5 \\ 2x + y = \sqrt[3]{9} \end{cases}$$

Possible Qualitative Solution(s) to Linear Systems

Definition

(Solution of a Linear System)

A **solution** of a $m \times n$ linear system satisfies all m linear eqn's simultaneously.

Theorem

(Qualitative Solution Possibilities to Linear Systems)

For a $m \times n$ linear system, exactly one of the following is true:

- *The linear system has a **unique solution** (i.e. one and only one solution)*
- *The linear system has **infinitely many solutions***
- *The linear system has **no solution***

Definition

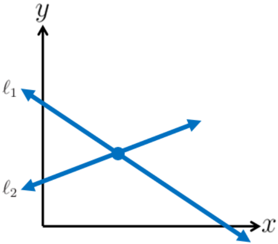
(Consistent & Inconsistent Linear Systems)

A $m \times n$ linear system is **inconsistent** $\iff$ it has **no solution**.

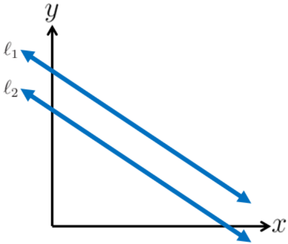
Otherwise, the linear system is called **consistent**.

Geometric Interpretation of 2×2 Linear Systems

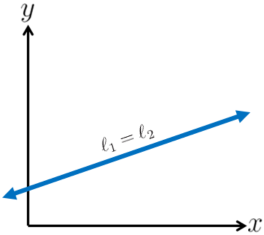
Prototype 2×2 Linear System: $\begin{cases} \ell_1 : & a_{11}x & + & a_{12}y & = & b_1 \\ \ell_2 : & a_{21}x & + & a_{22}y & = & b_2 \end{cases}$



Intersecting Lines
Square System
Unique Solution
Solution is a point



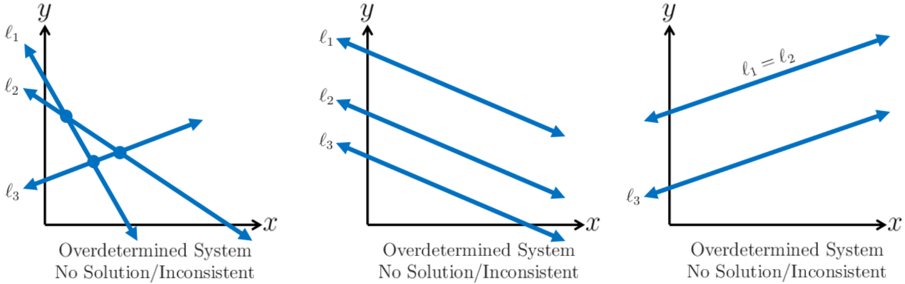
Parallel Lines
Square System
No Solution/Inconsistent



Coincident Lines
Square System
Infinitely Many Solutions
Solutions lie on line

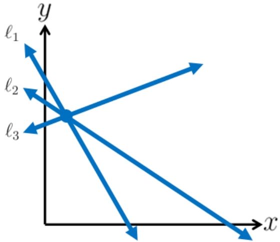
Prototype 3×2 Linear System:

$$\begin{cases} \ell_1 : & a_{11}x & + & a_{12}y & = & b_1 \\ \ell_2 : & a_{21}x & + & a_{22}y & = & b_2 \\ \ell_3 : & a_{31}x & + & a_{32}y & = & b_3 \end{cases}$$

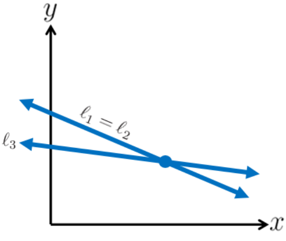


Geometric Interpretation of 3×2 Linear Systems

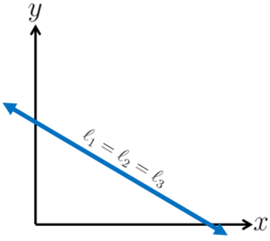
Prototype 3×2 Linear System:
$$\begin{cases} \ell_1 : & a_{11}x & + & a_{12}y & = & b_1 \\ \ell_2 : & a_{21}x & + & a_{22}y & = & b_2 \\ \ell_3 : & a_{31}x & + & a_{32}y & = & b_3 \end{cases}$$



Overdetermined System
Unique Solution
Solution is a point



Overdetermined System
Unique Solution
Solution is a point



Overdetermined System
Infinitely Many Solutions
Solutions lie on line

Geometric Interpretation of 3×3 Linear Systems

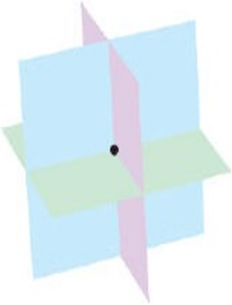
Prototype 3×3 Linear System:
$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$$

Intersecting Planes
Square System
No Solution/Inconsistent

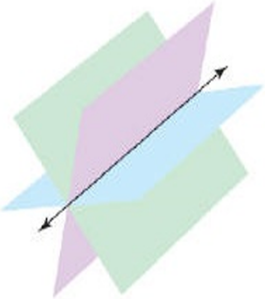
Parallel Planes
Square System
No Solution/Inconsistent

Geometric Interpretation of 3×3 Linear Systems

Prototype 3×3 Linear System:
$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$$



Intersecting Planes
Square System
Unique Solution
Solution is a point

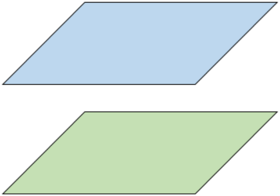


Intersecting Planes
Square System
Infinitely Many Solutions
Solutions lie on line

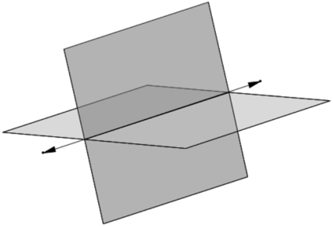
Coincident Planes
Square System
Infinitely Many Solutions
Solutions lie on plane

Geometric Interpretation of 2×3 Linear Systems

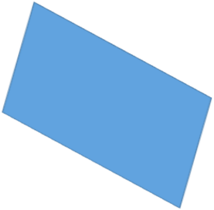
Prototype 2×3 Linear System: $\left\{ \begin{array}{rcrcrcl} a_{11}x & + & a_{12}y & + & a_{13}z & = & b_1 \\ a_{21}x & + & a_{22}y & + & a_{23}z & = & b_2 \end{array} \right.$



Parallel Planes
Underdetermined System
No Solution/Inconsistent



Intersecting Planes
Underdetermined System
Infinitely Many Solutions
Solutions lie on line



Coincident Planes
Underdetermined System
Infinitely Many Solutions
Solutions lie on plane

Geometric Interpretation of 4×4 Linear Systems

Prototype 4×4 Linear System:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z + a_{14}w = b_1 \\ a_{21}x + a_{22}y + a_{23}z + a_{24}w = b_2 \\ a_{31}x + a_{32}y + a_{33}z + a_{34}w = b_3 \\ a_{41}x + a_{42}y + a_{43}z + a_{44}w = b_4 \end{cases}$$

Hard to reliably visualize 4-dimensional **hyperplanes**!

Matrices, Row Vectors, Column Vectors (Definition)

Definition

(Matrix)
A $m \times n$ **matrix** is an array of scalars arranged in m rows & n columns.

Definition

(Square Matrix)
A **square matrix** is a matrix with as many rows as columns $(m = n)$.

Definition

(Row Vector)
A n**-wide row vector** is a $1 \times n$ matrix (i.e. only one row)

Definition

(Column Vector)
A m**-wide column vector** is a $m \times 1$ matrix (i.e. only one column)

Matrices, Row Vectors, Column Vectors (Examples)

2×3 Matrix: $\begin{bmatrix} 1 & 4 & 0 \\ -1 & \pi & \left(8 - 3\sqrt{5}\right) \end{bmatrix}$

3×2 Matrix: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 9 \end{bmatrix}$

2×2 Square Matrix: $\begin{bmatrix} \sqrt[3]{2} & 3/4 \\ 1 & 0 \end{bmatrix}$

4-wide Row Vector: $\begin{bmatrix} -3 & 7 & 0 & \sqrt{3} \end{bmatrix}$

2-wide Column Vector: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

3-wide Column Vector: $\begin{bmatrix} 7/5 \\ 2\pi \\ 1 \end{bmatrix}$

Writing Linear Systems in Matrix-Vector Form: $A\mathbf{x} = \mathbf{b}$

Proposition

(Matrix-Vector Form of a Linear System)

A $m \times n$ *linear system*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

can be written compactly in matrix-vector form as $A\mathbf{x} = \mathbf{b}$*, where*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A *is called the* ***coefficient matrix*** (A *is a* $m \times n$ *matrix*)
$\mathbf{x}$ *is called the* ***unknown vector*** ($\mathbf{x}$ *is a* n*-wide column vector*)
$\mathbf{b}$ *is called the* ***RHS vector*** ($\mathbf{b}$ *is a* m*-wide column vector*)

Writing Linear Systems as $A\mathbf{x} = \mathbf{b}$ (Examples)

$$\left\{ \begin{array}{rcrcl} x & - & y & = & 5 \\ 2x & + & y & = & \sqrt[3]{9} \end{array} \right. \qquad A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ \sqrt[3]{9} \end{bmatrix}$$

$$\left\{ \begin{array}{rcrcrcl} x & & & + & z & = & \pi \\ 3x & - & y & - & \frac{1}{2}z & = & 0 \end{array} \right. \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & -1 & -\frac{1}{2} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{b} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

$$\left\{ \begin{array}{rcrcl} x_1 & + & x_2 & = & 1 \\ \left(\sqrt{3}\right) x_1 & - & x_2 & = & \frac{3}{2} \\ x_1 & & & = & 0 \end{array} \right. \qquad A = \begin{bmatrix} 1 & 1 \\ \sqrt{3} & -1 \\ 1 & 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

Augmented Matrix (Definition)

Definition

(Augmented Matrix)

A $m \times n$ linear system

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right.$$

can be written compactly in matrix-vector form as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The system can be written more compactly as an **augmented matrix**: $[A \mid \mathbf{b}]$

i.e. Take coefficient matrix A, place vertical divider & augment RHS vector $\mathbf{b}$.

Augmented Matrix (Examples)

LINEAR SYSTEM:

CORRESPONDING AUGMENTED MATRIX:

$$\begin{cases} x - y = 5 \\ 2x + y = \sqrt[3]{9} \end{cases}$$

$$[\,A \mid \mathbf{b}\,] = \left[\begin{array}{cc|c} 1 & -1 & 5 \\ 2 & 1 & \sqrt[3]{9} \end{array}\right]$$

$$\begin{cases} x + z = \pi \\ 3x - y - \frac{1}{2}z = 0 \end{cases}$$

$$[\,A \mid \mathbf{b}\,] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & \pi \\ 3 & -1 & -\frac{1}{2} & 0 \end{array}\right]$$

$$\begin{cases} x_1 + x_2 = 1 \\ \left(\sqrt{3}\right)x_1 - x_2 = \frac{3}{2} \\ x_1 = 0 \end{cases}$$

$$[\,A \mid \mathbf{b}\,] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ \sqrt{3} & -1 & \frac{3}{2} \\ 1 & 0 & 0 \end{array}\right]$$

A Note about Vector Notation

For this course, a variable that represents a **vector** is always in **boldface**: $\mathbf{x}$

But some elementary books prefer to place an **arrow** above the variable: $\vec{\mathbf{x}}$

and many advanced books/research papers don't distinguish vectors at all: x

These slides & outlines will often just write vectors in **boldface**: $\mathbf{x}$

It's possible that homework/exam problems may bolden & place an arrow: $\vec{\mathbf{x}}$

When writing **by hand**, <u>optionally</u> place an arrow (no boldface): $\vec{x}$ OR x

Fin.