Solving $A\mathbf{x} = \mathbf{b}$: Gauss-Jordan Elimination

Linear Algebra

Josh Engwer

TTU

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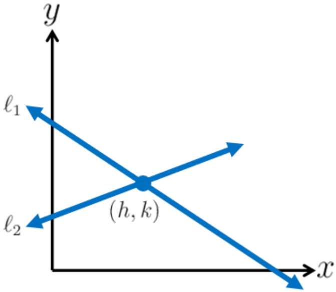
Definition

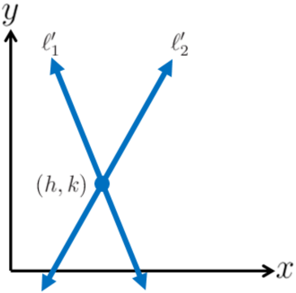
(Equivalent Linear Systems)

Two $m \times n$ linear systems are **equivalent** $\iff$ both have **same solution set**.

[LS-1]: $\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$ [LS-2]: $\begin{cases} c_{11}x + c_{12}y = d_1 \\ c_{21}x + c_{22}y = d_2 \end{cases}$



Graph of [LS-1]



Graph of [LS-2]

Two linear systems with exact same unique solution $(x, y) = (h, k)$

Equivalent Linear Systems (Definition)

Definition

(Equivalent Linear Systems)

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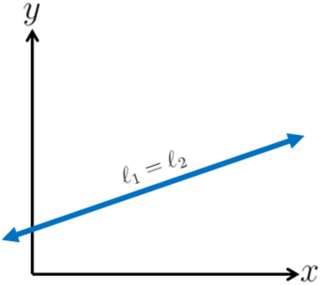
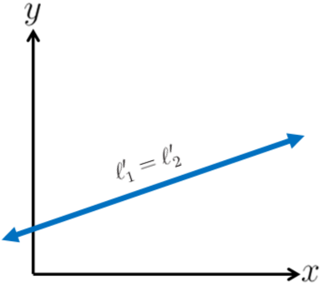


Graph of [LS-1]



Graph of [LS-2]

Two linear systems with exact same set of infinitely many solutions

Elementary Row Operations (Motivations)

Consider the following 2×2 **equivalent** linear systems:

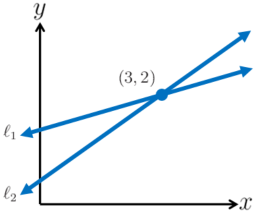
$$\text{[LS-1]: } \begin{cases} -x + 3y = 3 \\ x - 2y = -1 \end{cases} \iff [\,A \mid \mathbf{b}\,] = \left[\begin{array}{cc|c} -1 & 3 & 3 \\ 1 & -2 & -1 \end{array}\right]$$

$$\text{[LS-2]: } \begin{cases} -x + 3y = 3 \\ y = 2 \end{cases} \iff [\,A \mid \mathbf{b}\,] = \left[\begin{array}{cc|c} -1 & 3 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

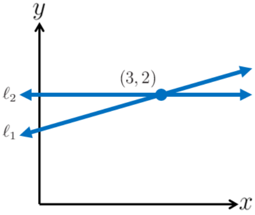
Both [LS-1] & [LS-2] have the exact same unique solution: $(x, y) = (3, 2)$

However, notice that [LS-2] is **simpler** to work with than [LS-1] due to:

- **Linear system** [LS-2] **having fewer unknowns present**.
- **Augmented matrix** for [LS-2] **having more zeros present in matrix** A.
- **(visually)** [LS-2] **having more lines perpendicular to coordinate axes**.



Graph of [LS-1]



Graph of [LS-2]

Elementary Row Operations (Definition)

When solving linear system $A\mathbf{x} = \mathbf{b}$, one should rewrite the system into a **simpler equivalent system**.

This can achieved using **elementary row operations** applied to the corresponding **augmented matrix** $[\, A \mid \mathbf{b} \,]$.

Definition

(Elementary Row Operations)

There are three types of **elementary row operations** applicable to $[\, A \mid \mathbf{b} \,]$:

(SWAP)	$[R_i \leftrightarrow R_j]$	Swap row i & row j
(SCALE)	$[\alpha R_j \to R_j]$	Multiply row j by a non-zero scalar α
(COMBINE)	$[\alpha R_i + R_j \to R_j]$	Add scalar multiple α of row i to row j

Elementary Row Operations (Examples)

(SWAP)
$$\left[\begin{array}{rr|r} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rr|r} 3 & 4 & 0 \\ 2 & 7 & 9 \\ -1 & 0 & 1 \end{array}\right]$$

(SCALE)
$$\left[\begin{array}{rr|r} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{array}\right] \xrightarrow{(-2)R_1 \rightarrow R_1} \left[\begin{array}{rr|r} (-2)(3) & (-2)(4) & (-2)(0) \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{array}\right]$$

(COMBINE)
$$\left[\begin{array}{rr|r} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{array}\right] \xrightarrow{3R_1 + R_3 \rightarrow R_3} \left[\begin{array}{rr|r} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 3(3)+2 & 3(4)+7 & 3(0)+9 \end{array}\right]$$

Elementary Row Operations (Examples)

(SWAP) $\left[\begin{array}{rr|r} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rr|r} 3 & 4 & 0 \\ 2 & 7 & 9 \\ -1 & 0 & 1 \end{array}\right]$

(SCALE) $\left[\begin{array}{rr|r} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{array}\right] \xrightarrow{(-2)R_1 \to R_1} \left[\begin{array}{rr|r} -6 & -8 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{array}\right]$

(COMBINE) $\left[\begin{array}{rr|r} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{array}\right] \xrightarrow{3R_1 + R_3 \to R_3} \left[\begin{array}{rr|r} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 11 & 19 & 9 \end{array}\right]$

Reduced Row-Echelon Form (RREF) of a Matrix

Question: When is linear system $A\mathbf{x} = \mathbf{b}$ its most simplest??
Answer: When the augmented matrix $[\ A \mid \mathbf{b}\ ]$ is in **RREF**.

Definition

(Reduced Row-Echelon Form (RREF) of a Matrix)

An matrix is in **reduced row-echelon form (RREF)** if the following are all true:

- Any rows consisting entirely of zeros occur below all non-zero rows.
- For each non-zero row, the first (**left-most**) **non-zero entry** is **1**. (such a 1 is called a **pivot** or **leading one**)
- For two successive non-zero rows, the pivot in the higher row is farther to the left than the pivot in lower row.
- Every column with a pivot has zeros above & below its pivot.
- For **linear systems**, the $(1,1)$-entry must be a **pivot**. [NOTATION: "(i,j)-entry" means "i^{th} row, j^{th} column"]

The RREF of an augmented matrix seemingly may have a "pivot" in the last column, but it's not really a pivot!
However, still zero out entries above & below such a "pivot" in the last column.

RREF of an Augmented Matrix (Examples)

Examples of augmented matrices in RREF (pivots are boxed):

$$\left[\begin{array}{cc|c} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & 5 \end{array}\right], \left[\begin{array}{cc|c} \boxed{1} & -3 & 0 \\ 0 & 0 & 1 \end{array}\right], \left[\begin{array}{cc|c} \boxed{1} & -3 & 4 \\ 0 & 0 & 0 \end{array}\right],$$

$$\left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \end{array}\right], \left[\begin{array}{cc|c} \boxed{1} & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|c} \boxed{1} & -3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right], \left[\begin{array}{ccc|c} \boxed{1} & 6 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \left[\begin{array}{ccc|c} \boxed{1} & 6 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & -1 & 3 \\ 0 & \boxed{1} & 0 & 7 & 5 \\ 0 & 0 & \boxed{1} & 1 & 1 \end{array}\right], \left[\begin{array}{cccc|c} \boxed{1} & 0 & 9 & 2 & 0 \\ 0 & \boxed{1} & 4 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right], \left[\begin{array}{cccc|c} \boxed{1} & 0 & 9 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Gauss-Jordan Elimination (Procedure)

Question: How to rewrite augmented matrix $[\, A \mid \mathbf{b} \,]$ into RREF??
Answer: Apply **Gauss-Jordan elimination** to $[\, A \mid \mathbf{b} \,]$.

Proposition

(Gauss-Jordan Elimination)

*(1) SWAP/SCALE/COMBINE to zero-out entries **below pivots, left-to-right**.*
*(2) SWAP/SCALE/COMBINE to zero-out entries **above pivots, right-to-left**.*

<u>WARNING:</u> Zeroing out entries without using this particular sequence of steps may cause earlier zeroed-out entries to become non-zero again!! By doing this, you are doing more work than is necessary!!

* indicates possibly non-zero entries

Pivots are boxed:

$$[\,A \mid \mathbf{b}\,] = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & 0 & * \\ 0 & \boxed{1} & * \end{array}\right] = \left[\,\text{RREF}(A) \,\middle|\, \widetilde{\mathbf{b}}\,\right]$$

$$[\,A \mid \mathbf{b}\,] = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & * & * \\ 0 & 0 & 0 \end{array}\right] = \left[\,\text{RREF}(A) \,\middle|\, \widetilde{\mathbf{b}}\,\right]$$

$$[\,A \mid \mathbf{b}\,] = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & * & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\,\text{RREF}(A) \,\middle|\, \widetilde{\mathbf{b}}\,\right]$$

Gauss-Jordan Elim. (3×2 Prototype Possibitlies)

* indicates possibly non-zero entries Pivots are boxed:

$$[\, A \mid \mathbf{b} \,] = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & 0 & * \\ 0 & \boxed{1} & * \\ 0 & 0 & 0 \end{array}\right] = \left[\, \text{RREF}(A) \,\middle|\, \widetilde{\mathbf{b}} \,\right]$$

$$[\, A \mid \mathbf{b} \,] = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\, \text{RREF}(A) \,\middle|\, \widetilde{\mathbf{b}} \,\right]$$

$$[\, A \mid \mathbf{b} \,] = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] = \left[\, \text{RREF}(A) \,\middle|\, \widetilde{\mathbf{b}} \,\right]$$

$$[\, A \mid \mathbf{b} \,] = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] = \left[\, \text{RREF}(A) \,\middle|\, \widetilde{\mathbf{b}} \,\right]$$

Protip: Delay the Onslaught of Fractions (Part 1)

Sometimes an augmented matrix may have **fractions** in some entries:

$$\left[\begin{array}{cc|c} 3 & 1/2 & 1/5 \\ 1/3 & 2 & 4 \end{array}\right] \xrightarrow{\left(\frac{1}{3}\right)R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & 1/6 & 1/15 \\ 1/3 & 2 & 4 \end{array}\right]$$

This will cause Gauss-Jordan to involve tedious fraction arithmetic!

To avoid dealing with fractions (at least for a few steps),
SCALE each row with fractions by its **common denominator**:

$$\left[\begin{array}{cc|c} 3 & 1/2 & 1/5 \\ 1/3 & 2 & 4 \end{array}\right] \xrightarrow[3R_2 \to R_2]{10R_1 \to R_1} \left[\begin{array}{cc|c} 30 & 5 & 2 \\ 1 & 6 & 12 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} \boxed{1} & 6 & 12 \\ 30 & 5 & 2 \end{array}\right]$$

Protip: Delay the Onslaught of Fractions (Part 2)

Sometimes a SCALE to create a pivot may cause **fractions** in other entries:

$$\left[\begin{array}{cc|c} 3 & 4 & 8 \\ 2 & 3 & 0 \\ 5 & 6 & 3 \end{array}\right] \xrightarrow{\left(\frac{1}{3}\right)R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & \mathbf{4/3} & \mathbf{8/3} \\ 2 & 3 & 0 \\ 5 & 6 & 3 \end{array}\right]$$

This will cause later Gauss-Jordan steps to involve tedious fraction arithmetic!

To avoid dealing with fractions (at least for a few steps),
Zero-out each entry below a would-be pivot by **SCALE**-ing **each pair of rows** such that the two entries are **identical**:

$$\left[\begin{array}{cc|c} \mathbf{3} & 4 & 8 \\ \mathbf{2} & 3 & 0 \\ 5 & 6 & 3 \end{array}\right] \xrightarrow[3R_2 \to R_2]{2R_1 \to R_1} \left[\begin{array}{cc|c} \mathbf{6} & 8 & 16 \\ \mathbf{6} & 9 & 0 \\ 5 & 6 & 3 \end{array}\right] \xrightarrow{(-1)R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 6 & 8 & 16 \\ \mathbf{0} & 1 & -16 \\ 5 & 6 & 3 \end{array}\right] \xrightarrow{\left(\frac{1}{2}\right)R_1 \to R_1}$$

$$\left[\begin{array}{cc|c} \mathbf{3} & 4 & 8 \\ \mathbf{0} & 1 & -16 \\ \mathbf{5} & 6 & 3 \end{array}\right] \xrightarrow[3R_3 \to R_3]{5R_1 \to R_1} \left[\begin{array}{cc|c} \mathbf{15} & 20 & 40 \\ \mathbf{0} & 1 & -16 \\ \mathbf{15} & 18 & 9 \end{array}\right] \xrightarrow{(-1)R_1 + R_3 \to R_3} \left[\begin{array}{cc|c} 15 & 20 & 40 \\ \mathbf{0} & 1 & -16 \\ \mathbf{0} & -2 & -31 \end{array}\right]$$

$$\xrightarrow{\left(\frac{1}{15}\right)R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & \mathbf{4/3} & \mathbf{8/3} \\ \mathbf{0} & 1 & -16 \\ \mathbf{0} & -2 & -31 \end{array}\right]$$

(Again, fractions may be inevitable, but at least they can be delayed. NOTE: entries may become **quite large**!)

Solving $A\mathbf{x} = \mathbf{b}$ using Gauss-Jordan Elimination

Definition

A mathematical statement is a **tautology** $\iff$ it is **always true**.
A mathematical statement is a **contradiction** $\iff$ it is **always false**.

TAUTOLOGY:	**CONTRADICTION:**
$3 = 3$	$0 = 1$
7 is a prime number	6 is a prime number
If $f(x) = x^2$, then $f'(x) = 2x$	If $f(x) = \log x$, then $f'(x) = 0$

If a row of $[\ \text{RREF}(A) \mid \widetilde{\mathbf{b}}\ ]$ translates to a **TAUTOLOGY** $(0 = 0)$, then proceed as usual. (Unique soln, no soln, or infinitely many soln's can occur)

If a row of $[\ \text{RREF}(A) \mid \widetilde{\mathbf{b}}\ ]$ translates to a **CONTRADICTION**, then linear system $A\mathbf{x} = \mathbf{b}$ has **no solution**:

$$\left[\begin{array}{cc|c} \boxed{1} & 3 & 0 \\ 0 & 0 & 1 \end{array}\right] \iff \begin{cases} x_1 + 3x_2 = 0 \\ 0 = 1 \leftarrow \text{Contradiction!} \end{cases} \quad \therefore \boxed{\text{No Soln}}$$

$$\left[\begin{array}{cc|c} \boxed{1} & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \iff \begin{cases} x_1 + 3x_2 = 0 \\ 0 = 1 \leftarrow \text{Contradiction!} \\ 0 = 0 \leftarrow \text{Tautology} \end{cases} \quad \therefore \boxed{\text{No Soln}}$$

Solving $A\mathbf{x} = \mathbf{b}$ using Gauss-Jordan Elimination

Each column of RREF(A) that contains **a pivot** means corresponding unknown variable is a **fixed variable.**
Each column of RREF(A) that contains **no pivot** means corresponding unknown variable is a **free variable.**

Each **free variable** can be any scalar, so assign each one a **parameter.**
Each **fixed variable** must be expressed in terms of the **parameters.**

$$\left[\begin{array}{cc|c} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & 5 \end{array}\right] \implies$$

x_1, x_2 are fixed variables
There are no free variables

$$\therefore \boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 6 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies$$

x_1 is a fixed variable
x_2, x_3 are free variables
Let $x_2 = s$ and $x_3 = t$, where parameters $s, t \in \mathbb{R}$
Then $x_1 + 6x_2 + 2x_3 = 3 \implies x_1 = 3 - 6s - 2t$

Column Vector Form:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 6s - 2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}$$

Tuple Form: $\boxed{(x_1, x_2, x_3) = (3 - 6s - 2t, s, t)}$

Solving $A\mathbf{x} = \mathbf{b}$ via Gauss-Jordan Elim. (Procedure)

Proposition

(Solving $m \times n$ Linear System $A\mathbf{x} = \mathbf{b}$ using Gauss-Jordan Elimination)

*(1) Form **augmented matrix** $[\, A \mid \mathbf{b} \,]$*
*(2) SWAP/SCALE/COMBINE to zero-out entries **below pivots, left-to-right***
*(3) SWAP/SCALE/COMBINE to zero-out entries **above pivots, right-to-left***

At this point, Gauss-Jordan is done: $[\, A \mid \mathbf{b} \,] \xrightarrow{Gauss-Jordan} [\, \mathit{RREF}(A) \mid \widetilde{\mathbf{b}} \,]$

(4) Translate augmented matrix $[\, \mathit{RREF}(A) \mid \widetilde{\mathbf{b}} \,]$ into equivalent linear system:

*If any equations are a **CONTRADICTION**, then system has **no solution***
*If any equations are a **TAUTOLOGY**, then proceed as usual to STEP (5)*

*(5) Identify all **fixed variables** & **free variables** using RREF(A):*

*Each column that contains **a pivot** corresponds to a **fixed variable***
*Each column that contains **no pivot** corresponds to a **free variable***

*(6) Assign each **free variable** a unique **parameter***
*(7) Express each **fixed variable** in terms of the **parameters***
*(8) Write out **solution** in either **tuple form** or **column vector form***

Why no mention of Gaussian Elimination??

You may have learned Gaussian Elimination from a previous course.
For most of this course, Gauss-Jordan Elimination is absolutely essential.

Gaussian Elimination is only used in this course for:

LU-Factorization of a **Square** Matrix	(Larson 2.4)
Determinant of a large dense **Square** Matrix	(Larson 3.2)

So, we'll encounter it at these sections of the textbook, but I'll describe the procedure more as "Elementary Row Operations to achieve a certain structure in the matrix" rather than "Gaussian Elimination."

Gaussian Elim. has a central role in Numerical Linear Algebra. (MATH 4312)

Fin.