Matrix Algebra: Add, Transpose, Multiply

Linear Algebra

Josh Engwer

TTU

02 September 2015

PART I:

MATRIX ADDITION, MATRIX SUBTRACTION, SCALAR MULTIPLICATION, TRANSPOSES

Compact Notation for Arbitrary Matrix Entries

It's too tedious to represent the entries of an arbitrary matrix as shown before. Fortunately, there's a far simpler conventional notation:

Proposition

(Compact Notation for Arbitrary Matrix Entries)

An arbitrary $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

can be represented compactly as

$$A = [a_{ij}]_{m\times n} \qquad (\textit{where } 1 \le i \le m \; \textit{ and } \; 1 \le j \le n)$$

If the ***shape*** *of A is not known a priori, then simply write:* $A = [a_{ij}]$

Equality of Matrices (Definition)

Question: When are two matrices equal??

Definition

(Equal Matrices)

Let $A = [a_{ij}], B = [b_{ij}]$ be matrices of arbitrary shapes.
Then matrices A and B are **equal** $\iff$ the following are all true:

(i) Matrices A and B have the **same shape**: A and B are both $m \times n$
(ii) Their **corresponding entries** are **equal**: $a_{ij} = b_{ij} \ \forall i, j$

LOGIC NOTATION: The symbol $\forall$ means "for all", "for every"

Equality of Matrices (Examples)

$$\begin{bmatrix} 1 & 3 & \pi \\ \sqrt{2} & -1 & 7 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & x \\ \sqrt{2} & y & 7 \\ z & 0 & 1 \end{bmatrix} \iff \begin{cases} x = \pi \\ y = -1 \\ z = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$ since the $(1,3)$-entries are unequal: $3 \neq 0$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$ since LHS is a 3×2 matrix
RHS is a 2×3 matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$ since LHS is a 4-wide **row** vector
RHS is a 4-wide **column** vector

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \neq 5$$ since LHS is a 2×4 matrix
RHS is a scalar

Matrix Addition (Definition)

Adding a matrix to another matrix is defined as expected:

Definition

(Matrix Addition)

Let matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$. Then:

$$A + B := [a_{ij} + b_{ij}]_{m\times n}$$

i.e. The (i,j)-entry of $A+B$ is the sum of the (i,j)-entries of A and B.

NOTE: $A+B$ is **undefined** if matrices A and B have **different shapes**.

Matrix Addition (Examples)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 3 & \pi \\ 7 & -1 & \sqrt{3} \\ 4 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & (3+\pi) \\ 11 & 4 & (6+\sqrt{3}) \\ 11 & 13 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$ is undefined.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$ is undefined.

$$\begin{bmatrix} 4 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$ is undefined.

$$\begin{bmatrix} 4 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$ is undefined.

$$5 + \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$ is undefined.

Scalar Multiplication of a Matrix (Definition)

Multiplying a matrix by a **scalar** is defined as expected:

Definition

(Scalar Multiplication of a Matrix)

Let matrix $A = [a_{ij}]_{m \times n}$ and scalar $\alpha \in \mathbb{R}$. Then:

$$\alpha A := [\alpha a_{ij}]_{m \times n}$$

i.e. The (i,j)-entry of αA is the product of α and the (i,j)-entry of A.

NOTE: αA is **never undefined**.

Scalar Multiplication of a Matrix (Examples)

$$(2)\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} (2)(1) & (2)(2) \\ (2)(3) & (2)(4) \\ (2)(5) & (2)(6) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$(-1)\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} (-1)(1) & (-1)(2) & (-1)(3) \\ (-1)(4) & (-1)(5) & (-1)(6) \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \end{bmatrix}$$

$$(4)\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} (4)(1) & (4)(2) & (4)(3) \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \end{bmatrix}$$

$$\left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}\right)(1) \\ \left(\frac{1}{2}\right)(2) \\ \left(\frac{1}{2}\right)(3) \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 3/2 \end{bmatrix}$$

Matrix Subtraction (Definition)

Matrix subtraction is formally defined in terms of matrix addition & scalar mult.:

Definition

(Matrix Subtraction)

Let matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$. Then:

$$A - B := A + (-1)B$$

Of course in practice, matrix subtraction is performed as expected:

Corollary

(Matrix Subtraction)

Let matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$. Then:

$$A - B := [a_{ij} - b_{ij}]_{m\times n}$$

i.e. The (i,j)-entry of $A-B$ is the difference of the (i,j)-entry of A by that of B.

*NOTE: $A-B$ is **undefined** if matrices A and B have **different shapes**.*

Matrix Subtraction (Examples)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -\pi \\ 7 & -1 & \sqrt{3} \\ 4 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & (3+\pi) \\ -3 & 6 & (6-\sqrt{3}) \\ 3 & 3 & 8 \end{bmatrix}$$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is undefined.

$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is undefined.

$\begin{bmatrix} 4 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$ is undefined.

$\begin{bmatrix} 4 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$ is undefined.

$5 - \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$ is undefined.

Partitioning a Matrix into Row/Column Vectors

Sometimes it's useful to **partition** a matrix in terms of:

Column Vectors: $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ | & | & | & | \end{bmatrix}$$

Row Vectors: $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \text{——} & \mathbf{u}_1 & \text{——} \\ \text{——} & \mathbf{u}_2 & \text{——} \\ \text{——} & \mathbf{u}_3 & \text{——} \end{bmatrix}$$

where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix}$

and $\mathbf{u}_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 9 & 0 & 1 & 2 \end{bmatrix}$

Such partitioning will be occasionally used later in the course.
Partitioning into **column vectors** will usually be preferred over row vectors.

Partitioning a Matrix into Blocks

Sometimes it's useful to **partition** a matrix in terms of **blocks**:

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 0 & 1 & 2 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

where **blocks**

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}, A_{12} = \begin{bmatrix} 3 & 4 \\ 7 & 8 \end{bmatrix}, A_{21} = \begin{bmatrix} 9 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

Such partitioning will <u>never</u> be used in this course.

However, it is seen in **Numerical Analysis** as it's critical in designing & analyzing numerical algorithms.

Transpose of a Matrix (Definition)

Definition

(Transpose of a Matrix)

Let matrix $A = [a_{ij}]_{m \times n}$. Then $A^T := [a_{ji}]_{n \times m}$

Notice that the transpose of a $m \times n$ matrix is a $n \times m$ matrix.

i.e. The k^{th} **row** of A becomes the k^{th} **column** of A^T.

i.e. The k^{th} **column** of A becomes the k^{th} **row** of A^T.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, then $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, then $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$

Transpose of a Scalar

Since a scalar is a 1×1 matrix with only one entry, its transpose is itself:

Corollary

(Transpose of a Scalar)

The transpose of a scalar is itself:

$$\textit{Let } \alpha \in \mathbb{R} \textit{ be a scalar.} \quad \textit{Then } \alpha^T = \alpha$$

e.g. $5^T = 5, (-2)^T = -2, \pi^T = \pi, (\sqrt{3})^T = \sqrt{3}, \left(\frac{1}{6}\right)^T = \frac{1}{6}, (2 - \sqrt{7})^T = 2 - \sqrt{7}$

Transpose of a Column Vector or Row Vector

A m-wide column vector is a $m \times 1$ matrix.
A n-wide row vector is a $1 \times n$ matrix.

Corollary

(Transpose of a Column or Row Vector)

The transpose of a m-wide ***column vector*** *is a m-wide* ***row vector****.*

The transpose of a n-wide ***row vector*** *is a n-wide* ***column vector****.*

e.g. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \implies \mathbf{u}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$

e.g. $\mathbf{v} = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} \implies \mathbf{v}^T = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$

Transpose of a Column Vector or Row Vector

A m-wide column vector is a $m \times 1$ matrix.
A n-wide row vector is a $1 \times n$ matrix.

Corollary

(Transpose of a Column or Row Vector)

*The transpose of a m-wide **column vector** is a m-wide **row vector**.*

*The transpose of a n-wide **row vector** is a n-wide **column vector**.*

Going forward after this section, some new conventions:

- Vectors will no longer have arrows above them (to avoid clutter)

  e.g. $\mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ instead of $\vec{\mathbf{u}} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$

- **Column** vectors will mostly be used – if a row vector's needed, transpose.

  e.g. $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \iff \mathbf{v}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ instead of $\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

PART II:

MATRIX MULTIPLICATION

Matrix Multiplication (Definition)

Definition

(Matrix Multiplication)

Let matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$. Then:

$$AB := \left[\sum_{k=1}^{n} a_{ik}b_{kj}\right]_{m \times p}$$

where the summation $\sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$
For AB to be well-defined, the 'middle dimensions' of A & B must match (n)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} \ddots & \vdots & \vdots \\ \cdots & c_{ij} & \cdots \\ \cdots & \vdots & \ddots \end{bmatrix}$$

where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

Examples where Matrix Multiplication is Undefined

The following matrix products are **undefined** since the 'middle dimensions' (in **red**) do not match:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{2\times {\color{red}3}} \underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}}_{{\color{red}2}\times 3}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{2\times {\color{red}3}} \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}}_{{\color{red}4}\times 2}$$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}}_{4\times {\color{red}2}} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{{\color{red}3}\times 3}$$

Matrix Multiplication (Step-by-Step)

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \end{bmatrix}}_{2\times 4} = \underbrace{\begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}}_{3\times 4}$$

Matrix Multiplication (Step-by-Step)

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} \mathbf{0} & 1 & 2 & 3 \\ \mathbf{4} & 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} \mathbf{8} & & & \\ & & & \\ & & & \end{bmatrix}$$

$$(\mathbf{1})(\mathbf{0}) + (\mathbf{2})(\mathbf{4}) = \mathbf{8}$$

Matrix Multiplication (Step-by-Step)

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{1} & 2 & 3 \\ 4 & \mathbf{5} & 6 & 7 \end{bmatrix} = \begin{bmatrix} 8 & \mathbf{11} & & \\ & & & \\ & & & \end{bmatrix}$$

$$(\mathbf{1})(\mathbf{1}) + (\mathbf{2})(\mathbf{5}) = \mathbf{11}$$

Matrix Multiplication (Step-by-Step)

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 & \mathbf{2} & 3 \\ 4 & 5 & \mathbf{6} & 7 \end{bmatrix} = \begin{bmatrix} 8 & 11 & \mathbf{14} & \\ & & & \\ & & & \end{bmatrix}$$

$$(\mathbf{1})(\mathbf{2}) + (\mathbf{2})(\mathbf{6}) = \mathbf{14}$$

Matrix Multiplication (Step-by-Step)

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & \mathbf{3} \\ 4 & 5 & 6 & \mathbf{7} \end{bmatrix} = \begin{bmatrix} 8 & 11 & 14 & \mathbf{17} \\ & & & \\ & & & \end{bmatrix}$$

$$(\mathbf{1})(\mathbf{3}) + (\mathbf{2})(\mathbf{7}) = \mathbf{17}$$

Matrix Multiplication (Step-by-Step)

$$\begin{bmatrix} 1 & 2 \\ \mathbf{3} & \mathbf{4} \\ 5 & 6 \end{bmatrix} \begin{bmatrix} \mathbf{0} & 1 & 2 & 3 \\ \mathbf{4} & 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 14 & 17 \\ \mathbf{16} & & & \\ & & & \end{bmatrix}$$

$$(\mathbf{3})(\mathbf{0}) + (\mathbf{4})(\mathbf{4}) = \mathbf{16}$$

Matrix Multiplication (Step-by-Step)

$$\begin{bmatrix} 1 & 2 \\ \mathbf{3} & \mathbf{4} \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{1} & 2 & 3 \\ 4 & \mathbf{5} & 6 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 14 & 17 \\ 16 & \mathbf{23} & & \\ & & & \end{bmatrix}$$

$$(\mathbf{3})(\mathbf{1}) + (\mathbf{4})(\mathbf{5}) = \mathbf{23}$$

Matrix Multiplication (Step-by-Step)

$$\begin{bmatrix} 1 & 2 \\ \mathbf{3} & \mathbf{4} \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 & \mathbf{2} & 3 \\ 4 & 5 & \mathbf{6} & 7 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 14 & 17 \\ 16 & 23 & \mathbf{30} & \\ & & & \end{bmatrix}$$

$$(\mathbf{3})(\mathbf{2}) + (\mathbf{4})(\mathbf{6}) = \mathbf{30}$$

Matrix Multiplication (Step-by-Step)

$$\begin{bmatrix} 1 & 2 \\ \mathbf{3} & \mathbf{4} \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & \mathbf{3} \\ 4 & 5 & 6 & \mathbf{7} \end{bmatrix} = \begin{bmatrix} 8 & 11 & 14 & 17 \\ 16 & 23 & 30 & \mathbf{37} \\ & & & \end{bmatrix}$$

$$(\mathbf{3})(\mathbf{3}) + (\mathbf{4})(\mathbf{7}) = \mathbf{37}$$

Matrix Multiplication (Step-by-Step)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ \mathbf{5} & \mathbf{6} \end{bmatrix} \begin{bmatrix} \mathbf{0} & 1 & 2 & 3 \\ \mathbf{4} & 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 14 & 17 \\ 16 & 23 & 30 & 37 \\ \mathbf{24} & & & \end{bmatrix}$$

$$(\mathbf{5})(\mathbf{0}) + (\mathbf{6})(\mathbf{4}) = \mathbf{24}$$

Matrix Multiplication (Step-by-Step)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ \mathbf{5} & \mathbf{6} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{1} & 2 & 3 \\ 4 & \mathbf{5} & 6 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 14 & 17 \\ 16 & 23 & 30 & 37 \\ 24 & \mathbf{35} & & \end{bmatrix}$$

$$(\mathbf{5})(\mathbf{1}) + (\mathbf{6})(\mathbf{5}) = \mathbf{35}$$

Matrix Multiplication (Step-by-Step)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ \mathbf{5} & \mathbf{6} \end{bmatrix} \begin{bmatrix} 0 & 1 & \mathbf{2} & 3 \\ 4 & 5 & \mathbf{6} & 7 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 14 & 17 \\ 16 & 23 & 30 & 37 \\ 24 & 35 & \mathbf{46} & \end{bmatrix}$$

$$(\mathbf{5})(\mathbf{2}) + (\mathbf{6})(\mathbf{6}) = \mathbf{46}$$

Matrix Multiplication (Step-by-Step)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ \mathbf{5} & \mathbf{6} \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & \mathbf{3} \\ 4 & 5 & 6 & \mathbf{7} \end{bmatrix} = \begin{bmatrix} 8 & 11 & 14 & 17 \\ 16 & 23 & 30 & 37 \\ 24 & 35 & 46 & \mathbf{57} \end{bmatrix}$$

$$(\mathbf{5})(\mathbf{3}) + (\mathbf{6})(\mathbf{7}) = \mathbf{57}$$

Multiplying a Row/Col Vector by a Row/Col Vector

$$\text{Let } \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}. \quad \text{Then:}$$

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = (1)(5) + (2)(6) + (3)(7) = 38$$

$$\mathbf{v}\mathbf{u}^T = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} (5)(1) & (5)(2) & (5)(3) \\ (6)(1) & (6)(2) & (6)(3) \\ (7)(1) & (7)(2) & (7)(3) \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 6 & 12 & 18 \\ 7 & 14 & 21 \end{bmatrix}$$

$$\mathbf{v}^T\mathbf{v} = \begin{bmatrix} 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = (5)(5) + (6)(6) + (7)(7) = 110$$

$$\mathbf{v}\mathbf{v}^T = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} (5)(5) & (5)(6) & (5)(7) \\ (6)(5) & (6)(6) & (6)(7) \\ (7)(5) & (7)(6) & (7)(7) \end{bmatrix} = \begin{bmatrix} 25 & 30 & 35 \\ 30 & 36 & 42 \\ 35 & 42 & 49 \end{bmatrix}$$

$\mathbf{uv}$, $\mathbf{vu}$, $\mathbf{v}^T\mathbf{u}^T$, $\mathbf{u}^T\mathbf{v}^T$ are all **undefined**.

Computing Col Vector × Row Vector via Scalar Mult.

Now, a column vector × a row vector (AKA **outer product**) is the trickiest!
It's less error-prone to use partitioning & scalar multiplication as shown below:

$$\text{Let } \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}. \quad \text{Then}$$

$$\mathbf{v}\mathbf{u}^T = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} | & | & | \\ (1)\mathbf{v} & (2)\mathbf{v} & (3)\mathbf{v} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 6 & 12 & 18 \\ 7 & 14 & 21 \end{bmatrix}$$

$$\mathbf{v}\mathbf{v}^T = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} | & | & | \\ (5)\mathbf{v} & (6)\mathbf{v} & (7)\mathbf{v} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 25 & 30 & 35 \\ 30 & 36 & 42 \\ 35 & 42 & 49 \end{bmatrix}$$

$$\mathbf{u}\mathbf{u}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} | & | & | \\ (1)\mathbf{u} & (2)\mathbf{u} & (3)\mathbf{u} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} | & | & | \\ (5)\mathbf{u} & (6)\mathbf{u} & (7)\mathbf{u} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 \\ 10 & 12 & 14 \\ 15 & 18 & 21 \end{bmatrix}$$

Matrix Multiplication (Examples)

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \end{bmatrix}}_{2\times 4} = \underbrace{\begin{bmatrix} 8 & 11 & 14 & 17 \\ 16 & 23 & 30 & 37 \\ 24 & 35 & 46 & 57 \end{bmatrix}}_{3\times 4}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}}_{2\times 2} = \underbrace{\begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}}_{2\times 2}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 7 \\ 8 \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix}}_{3\times 1} \qquad \underbrace{\begin{bmatrix} 7 & 8 & 9 \end{bmatrix}}_{1\times 3} \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{3\times 2} = \underbrace{\begin{bmatrix} 76 & 100 \end{bmatrix}}_{1\times 2}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}}_{1\times 4} \underbrace{\begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}}_{4\times 1} = 70 \qquad \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{4\times 1} \underbrace{\begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix}}_{1\times 4} = \underbrace{\begin{bmatrix} 5 & 6 & 7 & 8 \\ 10 & 12 & 14 & 16 \\ 15 & 18 & 21 & 24 \\ 20 & 24 & 28 & 32 \end{bmatrix}}_{4\times 4}$$

Matrix Multiplication is NOT Commutative in general!

Most of the time, **square** matrices do not commute:

Proposition

(Non-Commutativity of Matrix Multiplication)

*Let A and B both be **square** matrices.*
Then, in general, $AB \neq BA$.

Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Then:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

$$\therefore \quad AB \neq BA$$

Matrix Multiplication is NOT Commutative in general!

Of course, occasionally there are exceptions:

Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix}$. Then:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 6 & 10 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 6 & 10 \end{bmatrix}$$

$\therefore \quad AB = BA$ (in this particular instance!)

Fin.