Matrix Algebra: Properties, Identity & Zero Matrices

Linear Algebra

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Properties of Scalars (Review)

Recall from College Algebra the properties of scalars:

Theorem

(Properties of Scalars)

Let $a, b, c \in \mathbb{R}$ be scalars.
Then:

(*S1*)	$a + b = b + a$	*Commutativity of Scalar Addition*
(*S2*)	$a + (b + c) = (a + b) + c$	*Associativity of Scalar Addition*
(*S3*)	$a + 0 = a$	*Zero is Scalar Additive Identity*
(*S4*)	$a + (-a) = 0$	$-a$ *is Scalar Additive Inverse*
(*S5*)	$ab = ba$	*Commutativity of Ordinary Multiplication*
(*S6*)	$a(bc) = (ab)c$	*Associativity of Ordinary Multiplication*
(*S7*)	$(1)a = a$	*One is Ordinary Multiplicative Identity*
(*S8*)	$a^{-1}a = aa^{-1} = 1$	a^{-1} *is Ordinary Multiplicative Inverse*
(*S9*)	$a(b + c) = ab + ac$	*Distributing Ordinary Mult. over Scalar Add.*

The Zero Matrix (Definition)

Definition

(Zero Matrix)

The $m \times n$ matrix O is called the $m \times n$ **zero matrix** if

$$O = [0]_{m \times n}$$

i.e. **Every entry** of the zero matrix is **zero**.

Note that the zero matrix is denoted by a capital O, not the digit 0.

If the shape of the zero matrix is not inferred from context,
then write the shape in its subscript: $O_{m \times n}$

$$O_{1\times 2} = \begin{bmatrix} 0 & 0 \end{bmatrix}, O_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, O_{3\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \cdots$$

Properties of Matrix Addition & Scalar Mult.

Theorem

(Properties of Matrix Addition & Scalar Multiplication)

Let A, B, C *be* $m \times n$ *matrices,* O *be the* $m \times n$ *zero matrix, and* α, β *be scalars. Then:*

(*A1*)	$A + B = B + A$	*Commutativity of Matrix Addition*
(*A2*)	$A + (B + C) = (A + B) + C$	*Associativity of Matrix Addition*
(*A3*)	$A + O = A$	*Zero Matrix is Matrix Additive Identity*
(*A4*)	$A + (-A) = O$	$-A$ *is Matrix Additive Inverse*
(*A5*)	$(\alpha\beta)A = \alpha(\beta A)$	*Associativity of Scalar Multiplication*
(*A6*)	$(1)A = A$	*One is Scalar Multiplicative Identity*
(*A7*)	$\alpha(A + B) = \alpha A + \alpha B$	*Distributing Scalar Mult. over Matrix Add.*
(*A8*)	$(\alpha + \beta)A = \alpha A + \beta A$	*Distributing Scalar Mult. over Scalar Add.*

Properties of Matrix Addition & Scalar Mult.

Theorem

(Properties of Matrix Addition & Scalar Multiplication)

Let A, B, C be $m \times n$ matrices, O be the $m \times n$ zero matrix, and α, β be scalars. Then:

(*A1*)	$A + B = B + A$	*Commutativity of Matrix Addition*
(*A2*)	$A + (B + C) = (A + B) + C$	*Associativity of Matrix Addition*
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(*A7*)	$\alpha(A + B) = \alpha A + \alpha B$	*Distributing Scalar Mult. over Matrix Add.*
(*A8*)	$(\alpha + \beta)A = \alpha A + \beta A$	*Distributing Scalar Mult. over Scalar Add.*

PROOF:

(A1): $A + B = [a_{ij} + b_{ij}]_{m\times n} \overset{S1}{=} [b_{ij} + a_{ij}]_{m\times n} = B + A$ QED

Properties of Matrix Addition & Scalar Mult.

Theorem

(Properties of Matrix Addition & Scalar Multiplication)

Let A, B, C *be* $m \times n$ *matrices,* O *be the* $m \times n$ *zero matrix, and* α, β *be scalars. Then:*

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(*A7*)	$\alpha(A + B) = \alpha A + \alpha B$	*Distributing Scalar Mult. over Matrix Add.*
(*A8*)	$(\alpha + \beta)A = \alpha A + \beta A$	*Distributing Scalar Mult. over Scalar Add.*

PROOF:

(A4): $A + (-A) = [a_{ij} + (-a_{ij})]_{m \times n} \stackrel{S4}{=} [0]_{m \times n} = O$ QED

Kronecker Delta (Definition)

In higher math, physics & engineering, it's convenient to define the following:

Definition

(Kronecker Delta)

Let i, j be **positive integers**. Then the **Kronecker Delta** δ_{ij} is defined to be

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Here's how to interpret δ_{ij}:

- If $i = 1$ and $j = 1$, then $i = j \implies \delta_{ij} = 1$
- If $i = 3$ and $j = 3$, then $i = j \implies \delta_{ij} = 1$
- If $i = 2$ and $j = 3$, then $i \neq j \implies \delta_{ij} = 0$
- If $i = 1$ and $j = 4$, then $i \neq j \implies \delta_{ij} = 0$

The Identity Matrix (Definition)

Definition

(Identity Matrix)

The $n \times n$ **square** matrix I is called the $n \times n$ **identity matrix** if

$$I = [\delta_{ij}]_{n\times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

i.e. There are one's on the **main diagonal** ($i = j$) & zero's elsewhere ($i \neq j$).

If the shape of the identity matrix is not inferred from context, then write the shape in its subscript: $I_{n\times n}$ or I_n

$$I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \cdots$$

Properties of Matrix Multiplication & Identity Matrix

Theorem

(Properties of Matrix Multiplication)

Let A, B, C be matrices s.t. the given matrix products are well-defined. Moreover, let $\alpha \in \mathbb{R}$ be a scalar. Then:

(M1)	$A(BC) = (AB)C$	*Associativity of Matrix Multiplication*
(M2)	$\alpha(AB) = (\alpha A)B = A(\alpha B)$	*Factoring a Scalar from a Matrix Product*
(M3)	$A(B + C) = AB + AC$	*Distributing Matrix Mult. over Matrix Addition*
(M4)	$(B + C)A = BA + CA$	*Distributing Matrix Mult. over Matrix Addition*

<u>NOTE:</u> Since $AB \neq BA$ in general, $A(B + C) \neq (B + C)A$ in general.

Theorem

(Identity Matrix is the Matrix Multiplicative Identity)

Let A be a $m \times n$ matrix. Then: $AI_n = A$ and $I_m A = A$

*Let A be a $n \times n$ **square** matrix. Then: $AI_n = I_n A = A$*

Properties of Transposes

Theorem

(Properties of Transposes)

Let A, B *be matrices s.t. the given matrix sums/products are well-defined. Moreover, let* $\alpha \in \mathbb{R}$ *be a scalar. Then:*

(*T1*)	$(A^T)^T = A$	*Transpose of a Transpose is Invariant*
(*T2*)	$(A+B)^T = A^T + B^T$	*Transpose of a Matrix Sum*
(*T3*)	$(\alpha A)^T = \alpha(A^T)$	*Transpose of a Scalar Multiple*
(*T4*)	$(AB)^T = B^T A^T$	*Transpose of a Matrix Product*

Corollary

(Transpose of a Matrix Difference)

Let A, B *be* $m \times n$ *matrices. Then:* $(A-B)^T = A^T - B^T$

Properties of Transposes

Theorem

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Let A, B *be matrices s.t. the given matrix sums/products are well-defined. Moreover, let* $\alpha \in \mathbb{R}$ *be a scalar. Then:*

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Corollary

(Transpose of a Matrix Difference)

Let A, B *be* $m \times n$ *matrices. Then:* $(A-B)^T = A^T - B^T$

<u>PROOF:</u>

$$(A-B)^T = [A+(-1)B]^T \overset{T2}{=} A^T + [(-1)B]^T \overset{T3}{=} A^T + (-1)B^T = A^T - B^T$$ QED

Transpose of an Extended Matrix Sum/Product

Corollary

(Transpose of an Extended Matrix Sum)

Let A, B, C *be* $m \times n$ *matrices. Then:* $(A+B+C)^T = A^T + B^T + C^T$

Corollary

(Transpose of an Extended Matrix Sum)

Let A, B, C *be matrices s.t.* ABC *is well-defined. Then:* $(ABC)^T = C^T B^T A^T$

Transpose of an Extended Matrix Sum/Product

Corollary

(Transpose of an Extended Matrix Sum)

Let A, B, C *be* $m \times n$ *matrices. Then:* $(A+B+C)^T = A^T + B^T + C^T$

Corollary

(Transpose of an Extended Matrix Product)

Let A, B, C *be matrices s.t.* ABC *is well-defined. Then:* $(ABC)^T = C^T B^T A^T$

PROOF:

$(A+B+C)^T \stackrel{A2}{=} [(A+B)+C]^T \stackrel{T2}{=} (A+B)^T + C^T \stackrel{T2}{=} A^T + B^T + C^T$ QED

$(ABC)^T \stackrel{M1}{=} [(AB)C]^T \stackrel{T4}{=} C^T (AB)^T \stackrel{T4}{=} C^T B^T A^T$ QED

Powers of a Square Matrix (Definition)

Definition

(Power of a Square Matrix)

Let A be a $n \times n$ **square** matrix and I be the $n \times n$ **identity** matrix. Moreover, let k be a **positive integer**. Then:

$$\begin{aligned} A^0 &:= I \\ A^1 &:= A \\ A^2 &:= AA \\ A^3 &:= AAA \\ A^k &:= \underbrace{AA\cdots A}_{k \text{ factors}} \end{aligned}$$

Corollary

(Properties of Powers of a Square Matrix)

*Let A be a $n \times n$ **square** matrix and j, k be **nonnegative integers**. Then:*

$$(P1) \quad A^jA^k = A^{j+k} \qquad \text{and} \qquad (P2) \quad (A^j)^k = A^{jk}$$

Diagonal Matrix (Definition)

In some applications & higher math, it's desirable to have a **diagonal matrix**:

Definition

(Diagonal Matrix)

A $n \times n$ **square** matrix D is a **diagonal matrix** if

$$D = [d_{ij}\delta_{ij}]_{n\times n} = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

i.e. Possible non-zero's on **main diagonal** ($i = j$) & zero's elsewhere ($i \neq j$).

Diagonal Matrices: $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Powers of Diagonal Matrices

One benefit of diagonal matrices is powers are painless to find:

Proposition

(Powers of Diagonal Matrices)

Let D be a $n \times n$ ***diagonal matrix*** *and k be a* ***nonnegative integer****. Then:*

$$D^k = [d_{ij}^k \delta_{ij}]_{n\times n} = \begin{bmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \implies A^3 = \begin{bmatrix} 1^3 & 0 \\ 0 & 2^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \implies B^4 = \begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 0^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

The Value of Scalar Algebra

So why care about all these properties involving scalars???

- Simplification/Expansion of Scalar Expressions

$$\begin{aligned}
(a+b)^2 &= (a+b)(a+b) && [a, b \text{ are scalars}] \\
&= (a+b)a + (a+b)b && [\text{(S9)}] \\
&= aa + ba + ab + bb && [\text{(S9)}] \\
&= aa + ab + ab + bb && [\text{(S5)}] \\
&= a^2 + 2ab + b^2
\end{aligned}$$

- Solving Scalar Equations

$$\begin{aligned}
\text{Solve for } x: \quad & 5x - 8a + 7b = 0 \qquad [x, a, b \text{ are scalars}] \\
\iff \quad & 5x - 8a + 7b - 7b = 0 - 7b \\
\iff \quad & 5x - 8a + 0 = -7b \\
\iff \quad & 5x - 8a = -7b \\
\iff \quad & 5x - 8a + 8a = -7b + 8a \\
\iff \quad & 5x + 0 = -7b + 8a \\
\iff \quad & 5x = 8a - 7b \\
\iff \quad & \left(\tfrac{1}{5}\right)5x = \tfrac{1}{5}(8a - 7b) \\
\iff \quad & (1)x = \tfrac{8}{5}a - \tfrac{7}{5}b \\
\iff \quad & x = \tfrac{8}{5}a - \tfrac{7}{5}b
\end{aligned}$$

The Value of Matrix Algebra

So why care about all these properties involving matrices???

- Simplification/Expansion of Matrix Expressions

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B) && [A, B \text{ are } n \times n \textbf{ square} \text{ matrices}]\\ &= (A+B)A + (A+B)B && [\text{(M3)}]\\ &= AA + BA + AB + BB && [\text{(M4)}]\\ &= A^2 + BA + AB + B^2\end{aligned}$$

- Solving Matrix Equations

Solve for X : $5X - 8A + 7B = O$ $[X, A, B \text{ are } m \times n \text{ matrices}]$

$$\begin{aligned}\Longleftrightarrow\quad & 5X - 8A + 7B - 7B = O - 7B\\ \Longleftrightarrow\quad & 5X - 8A + O = -7B\\ \Longleftrightarrow\quad & 5X - 8A = -7B\\ \Longleftrightarrow\quad & 5X - 8A + 8A = -7B + 8A\\ \Longleftrightarrow\quad & 5X + O = -7B + 8A\\ \Longleftrightarrow\quad & 5X = 8A - 7B\\ \Longleftrightarrow\quad & \left(\tfrac{1}{5}\right)5X = \tfrac{1}{5}(8A - 7B)\\ \Longleftrightarrow\quad & (1)X = \tfrac{8}{5}A - \tfrac{7}{5}B\\ \Longleftrightarrow\quad & X = \tfrac{8}{5}A - \tfrac{7}{5}B\end{aligned}$$

Fin.