Solving Square $A\mathbf{x} = \mathbf{b}$: Inverse Matrix

Linear Algebra

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11 September 2015

Properties of Scalars (Review)

Recall from College Algebra the properties of scalars:

Theorem

(Properties of Scalars)

Let $a, b, c \in \mathbb{R}$ be scalars.
Then:

(*S1*)	$a + b = b + a$	*Commutativity of Scalar Addition*
(*S2*)	$a + (b + c) = (a + b) + c$	*Associativity of Scalar Addition*
(*S3*)	$a + 0 = a$	*Zero is Scalar Additive Identity*
(*S4*)	$a + (-a) = 0$	$-a$ *is Scalar Additive Inverse*
(*S5*)	$ab = ba$	*Commutativity of Ordinary Multiplication*
(*S6*)	$a(bc) = (ab)c$	*Associativity of Ordinary Multiplication*
(*S7*)	$(1)a = a$	*One is Ordinary Multiplicative Identity*
(*S8*)	$a^{-1}a = aa^{-1} = 1$	a^{-1} *is Ordinary Multiplicative Inverse* $(a \neq 0)$
(*S9*)	$a(b + c) = ab + ac$	*Distributing Ordinary Mult. over Scalar Add.*

Solving Scalar Linear Equation $ax = b$ (Motivation)

Recall from College Algebra how to solve **scalar** linear eqn $ax = b$ for x:

CASE I: Suppose $a \neq 0$. Then:

$$\begin{array}{rcll} ax & = & b & \\ a^{-1}ax & = & a^{-1}b & \text{[Multiply both sides by } a^{-1}\text{]} \\ (a^{-1}a)x & = & a^{-1}b & \text{[S6]} \\ (1)x & = & a^{-1}b & \text{[S8]} \\ x & = & a^{-1}b & \text{[S7]} \end{array}$$

$\therefore$ $\boxed{x = a^{-1}b}$, where $a^{-1} = \dfrac{1}{a}$ is the multiplicative **inverse** of a.

CASE II: Suppose $a = 0$. Then:

$$ax = b \implies (0)x = b \implies 0 = b \implies \begin{cases} \text{Infinitely many solns} & \text{if } b = 0 \\ \text{No solution} & \text{if } b \neq 0 \end{cases}$$

Inverse of a Square Matrix (Definition)

Question: Is there an **inverse** of **matrix** A when solving linear sys $A\mathbf{x} = \mathbf{b}$?
Answer: Provided the linear system/matrix is **square**, then **maybe**:

Definition

(Inverse of a Square Matrix)

Let A be a $n \times n$ **square** matrix and I be the $n \times n$ **identity** matrix.
Then A is **invertible** if there exists a $n \times n$ matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I$$

A^{-1} is called the (matrix multiplicative) **inverse** of A.
If A does not have an inverse, A is called **singular** (AKA **noninvertible**).

Non-square matrices do not have inverses (since $AB \neq BA$ if $m \neq n$.)

Inverse of a Square Matrix (Uniqueness)

Question: Is it possible for an invertible matrix to have two or more inverses?
Answer: No!! There will be one and only one inverse:

Theorem

(Uniqueness of an Inverse Matrix)

If $n \times n$ square matrix A is invertible, then its inverse A^{-1} is ***unique***.

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Theorem

(Uniqueness of an Inverse Matrix)

If $n \times n$ square matrix A is invertible, then its inverse A^{-1} is ***unique***.

PROOF: Let A be a $n \times n$ **invertible** matrix and I be the $n \times n$ identity matrix.
Assume A has **two inverses**: A^{-1} and A_1^{-1}
Then by definition of inverse of A: $A^{-1}A = AA^{-1} = I$ and $A_1^{-1}A = AA_1^{-1} = I$

$$\begin{array}{rrcll} & AA_1^{-1} & = & I & \text{[Definition of Inverse of } A\text{]} \\ \Longrightarrow & A^{-1}AA_1^{-1} & = & A^{-1}I & \text{[Left-Multiply both sides by } A^{-1}\text{]} \\ \Longrightarrow & A^{-1}AA_1^{-1} & = & A^{-1} & [I \text{ is Matrix Multiplicative Identity}] \\ \Longrightarrow & (A^{-1}A)A_1^{-1} & = & A^{-1} & \text{[Associativity of Matrix Multiplication]} \\ \Longrightarrow & IA_1^{-1} & = & A^{-1} & \text{[Definition of Inverse of } A\text{]} \\ \Longrightarrow & A_1^{-1} & = & A^{-1} & [I \text{ is Matrix Multiplicative Identity}] \end{array}$$

$\therefore$ The two inverses A^{-1} and A_1^{-1} are actually the same.
$\therefore$ The inverse of A is unique. QED

How to Systematically find the Inverse of a Matrix?

Consider finding the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Then, if the inverse $A^{-1} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ exists:

$$AA^{-1} = I \implies \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\implies \begin{cases} x_{11} + 2x_{21} = 1 \\ 3x_{11} + 4x_{21} = 0 \end{cases} \text{ and } \begin{cases} x_{12} + 2x_{22} = 0 \\ 3x_{12} + 4x_{22} = 1 \end{cases}$$

$$\implies \text{ Perform Gauss-Jordan on } \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 0 \end{array}\right] \text{ and } \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 1 \end{array}\right]$$

$$\implies \text{ Perform Gauss-Jordan on } \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] = [A|I]$$

If A^{-1} exists, then linear systems have unique soln's $\implies$ RREF$(A) = I$.

If A is singular, then linear systems have no solution $\implies$ RREF$(A) \neq I$.

This analysis generalizes to when A is $n \times n$.

Finding an Inverse via Gauss-Jordan Elimination

Question: So how to find the inverse of A (if it exists)?
Answer: Apply **Gauss-Jordan Elimination** as follows:

Theorem

(Finding an Inverse via Gauss-Jordan Elimination)

GIVEN: ***Square*** *$n \times n$ matrix A.*

TASK: Find A^{-1} if it exists, otherwise conclude A is singular.

*(1) Form **augmented** matrix $[A|I]$, where I is $n \times n$ **identity** matrix.*
*(2) Apply **Gauss-Jordan Elimination** to $[A|I]$:*
*If RREF$(A) \neq I$, then A is **singular**.*
If RREF$(A) = I$, then $[A|I] \xrightarrow{Gauss-Jordan} [I|A^{-1}]$

SANITY CHECK: Check that $A^{-1}A = I$ and $AA^{-1} = I$.

Finding A^{-1} via Gauss-Jordan Elim. (Examples)

<u>WEX 2-3-1:</u> Using Gauss-Jordan, find the inverse (if it exists) of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{cc|cc} \boxed{1} & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{(-3)R_1+R_2\to R_2} \left[\begin{array}{cc|cc} \boxed{1} & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+R_1\to R_1} \left[\begin{array}{cc|cc} \boxed{1} & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \xrightarrow{\left(-\frac{1}{2}\right)R_2\to R_2} \left[\begin{array}{cc|cc} \boxed{1} & 0 & -2 & 1 \\ 0 & \boxed{1} & \frac{3}{2} & -\frac{1}{2} \end{array}\right] = [I|A^{-1}]$$

$$\therefore \quad \boxed{A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}}$$

Finding A^{-1} via Gauss-Jordan Elim. (Examples)

<u>WEX 2-3-2:</u> Using Gauss-Jordan, find the inverse (if it exists) of $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{cc|cc} \boxed{1} & 1 & 1 & 0 \\ 3 & 3 & 0 & 1 \end{array}\right] \xrightarrow{(-3)R_1+R_2 \to R_2} \left[\begin{array}{cc|cc} \boxed{1} & 1 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right] = [\text{RREF}(A)|B]$$

$\therefore$ Since $\text{RREF}(A) \neq I$, A^{-1} does <u>not</u> exist $\implies$ $\boxed{A \text{ is } \textbf{singular}}$

Inverse of a 2×2 Matrix

For 2×2 matrices, there's a simple formula to use to find an inverse:

Corollary

(Inverse of a 2×2 Matrix)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a 2×2 matrix s.t.* $a, b, c, d \in \mathbb{R}$. *Then:*

$$\text{If } ad - bc \neq 0, \text{ then } A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is ***singular****.*

PROOF: Apply Gauss-Jordan to augmented matrix $[A|I] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$.

Properties of Inverses

Theorem

(Properties of Inverse Matrices)

Let A, B *be* $n \times n$ ***invertible*** *matrices,* k *be* ***positive integer****, and* $\alpha \neq 0$*.*
Then A^{-1}*,* A^k*,* αA*,* A^T*,* AB *are all invertible and the following are true:*

(*I1*)	$(A^{-1})^{-1} = A$	*Inverse of an Inverse*
(*I2*)	$(A^k)^{-1} = (A^{-1})^k$	*Inverse of a Power*
(*I3*)	$(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$	*Inverse of a Scalar Mult.*
(*I4*)	$(A^T)^{-1} = (A^{-1})^T$	*Inverse of a Transpose*
(*I5*)	$(AB)^{-1} = B^{-1}A^{-1}$	*Inverse of a Product*

Properties of Inverses

Theorem

(Properties of Inverse Matrices)

*Let A, B be $n \times n$ **invertible** matrices, k be **positive integer**, and $\alpha \neq 0$. Then A^{-1}, A^k, αA, A^T, AB are all invertible and the following are true:*

(*I1*)	$(A^{-1})^{-1} = A$	*Inverse of an Inverse*
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(*I5*)	$(AB)^{-1} = B^{-1}A^{-1}$	*Inverse of a Product*

PROOF: Let I be the $n \times n$ identity matrix.

(I1): $A^{-1}A = AA^{-1} = I \implies A$ is inverse of $A^{-1} \implies (A^{-1})^{-1} = A$ QED

Properties of Inverses

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(Properties of Inverse Matrices)

*Let A, B be $n \times n$ **invertible** matrices, k be **positive integer**, and $\alpha \neq 0$. Then A^{-1}, A^k, αA, A^T, AB are all invertible and the following are true:*

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(I4)	$(A^T)^{-1} = (A^{-1})^T$	*Inverse of a Transpose*
(I5)	$(AB)^{-1} = B^{-1}A^{-1}$	*Inverse of a Product*

PROOF: Let I be the $n \times n$ identity matrix.

(I5):

$$\left.\begin{array}{l} (B^{-1}A^{-1})(AB) \overset{M1}{=} B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}B = I \\ (AB)(B^{-1}A^{-1}) \overset{M1}{=} A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I \end{array}\right. \implies (AB)^{-1} = B^{-1}A^{-1}$$

QED

Inverse of an Extended Product

How to find the **inverse** of a product of three matrices?

Corollary

(Inverse of an Extended Product)

*Let A, B, C be $n \times n$ **invertible** matrices. Then:*

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

PROOF: $(ABC)^{-1} \overset{M1}{=} [(AB)C]^{-1} \overset{I5}{=} C^{-1}(AB)^{-1} \overset{I5}{=} C^{-1}B^{-1}A^{-1}$ QED

This can be generalized to any matrix extended product:

Corollary

(Inverse of a Generalized Extended Product)

*Let $A_1, A_2, \ldots, A_{k-1}, A_k$ be $n \times n$ **invertible** matrices. Then:*

$$(A_1A_2\cdots A_{k-1}A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$$

Cancellation Properties of Matrix Products

Recall from College Algebra how to cancel a factor of a **scalar** product:

Let $a, b \in \mathbb{R}$ and $c \neq 0$. Then:

$$ac = bc \implies ac\left(\tfrac{1}{c}\right) = bc\left(\tfrac{1}{c}\right) \implies a(1) = b(1) \implies a = b$$
$$ca = cb \implies \left(\tfrac{1}{c}\right)ca = \left(\tfrac{1}{c}\right)cb \implies (1)a = (1)b \implies a = b$$

Question: Is there a similar cancelling behavior for **matrix products**?
Answer: **Yes**, provided the matrix to be cancelled is **invertible**:

Theorem

(Cancellation Properties of Matrix Products)

*Let C be an **invertible** matrix and A, B have compatible shapes. Then:*

(C1)	*If $AC = BC$, then $A = B$*	*Right-cancellation*
(C2)	*If $CA = CB$, then $A = B$*	*Left-cancellation*

Cancellation Properties of Matrix Products

Theorem

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$(C1)$	*If $AC = BC$, then $A = B$*	*Right-cancellation*
$(C2)$	*If $CA = CB$, then $A = B$*	*Left-cancellation*

PROOF: Since C is **invertible**, it has an inverse C^{-1} s.t. $C^{-1}C = CC^{-1} = I$ where I is the identity matrix with same shape as C.

$$\begin{array}{rrcll} & AC & = & BC & \text{[Given Statement]} \\ \Longrightarrow & ACC^{-1} & = & BCC^{-1} & \text{[Right-Multiply both sides by } C^{-1}\text{]} \\ \Longrightarrow & A(CC^{-1}) & = & B(CC^{-1}) & \text{[Associativity of Matrix Multiplication]} \\ \Longrightarrow & AI & = & BI & \text{[Definition of Inverse of } C\text{]} \\ \Longrightarrow & A & = & B & \text{[}I\text{ is Matrix Multiplicative Identity]} \end{array}$$

$\therefore$ If $AC = BC$, then $A = B$ QED

Cancellation of Matrix Products (WARNING)

Remember, for $AC = BC$ to imply $A = B$, C must be **invertible.**

Otherwise, it's possible for $AC = BC$ yet $A \neq B$:

Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 17 & 2 & -1 \\ 20 & 9 & 0 \\ 37 & 5 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 4 \\ 4 & -8 & 8 \end{bmatrix}$.

Then, clearly $A \neq B$ and yet

$$AC = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 4 \\ 4 & -8 & 8 \end{bmatrix} = \begin{bmatrix} 17 & -34 & 34 \\ 38 & -76 & 76 \\ 59 & -118 & 118 \end{bmatrix}$$

$$BC = \begin{bmatrix} 17 & 2 & -1 \\ 20 & 9 & 0 \\ 37 & 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 4 \\ 4 & -8 & 8 \end{bmatrix} = \begin{bmatrix} 17 & -34 & 34 \\ 38 & -76 & 76 \\ 59 & -118 & 118 \end{bmatrix}$$

Solving Square Linear System $A\mathbf{x} = \mathbf{b}$ via A^{-1}

How can A^{-1} be used to solve square linear system $A\mathbf{x} = \mathbf{b}$?

Theorem

(Square Linear Systems with Unique Solution)

*Let A be an **invertible** matrix.*
*Then square linear system $A\mathbf{x} = \mathbf{b}$ has **unique** solution given by $\mathbf{x} = A^{-1}\mathbf{b}$.*

REMARK: This is useful when solving several square linear systems with the same matrix A and different RHS $\mathbf{b}$'s since A^{-1} only has to be found once:

$$\begin{aligned} A\mathbf{x} = \mathbf{b}_1 &\implies \mathbf{x} = A^{-1}\mathbf{b}_1 \\ A\mathbf{x} = \mathbf{b}_2 &\implies \mathbf{x} = A^{-1}\mathbf{b}_2 \\ A\mathbf{x} = \mathbf{b}_3 &\implies \mathbf{x} = A^{-1}\mathbf{b}_3 \\ A\mathbf{x} = \mathbf{b}_4 &\implies \mathbf{x} = A^{-1}\mathbf{b}_4 \\ \vdots \quad & \qquad\quad \vdots \end{aligned}$$

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<u>PROOF:</u> Since A is **invertible**, it has inverse $A^{-1} \implies A^{-1}A = AA^{-1} = I$.
where I is the identity matrix with same shape as A.

$$
\begin{array}{rrcll}
 & A\mathbf{x} & = & \mathbf{b} & \text{[Given Statement]} \\
\implies & A^{-1}A\mathbf{x} & = & A^{-1}\mathbf{b} & \text{[Left-Multiply both sides by } A^{-1}\text{]} \\
\implies & (A^{-1}A)\mathbf{x} & = & A^{-1}\mathbf{b} & \text{[Associativity of Matrix Multiplication]} \\
\implies & (I)\mathbf{x} & = & A^{-1}\mathbf{b} & \text{[Definition of Inverse of } A\text{]} \\
\implies & \mathbf{x} & = & A^{-1}\mathbf{b} & [I \text{ is Matrix Multiplicative Identity]}
\end{array}
$$

Assume there are <u>two</u> solutions $\mathbf{x}_1$ and $\mathbf{x}_2$.
Then $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b} \implies \mathbf{x}_1 = A^{-1}\mathbf{b}$ and $\mathbf{x}_2 = A^{-1}\mathbf{b} \implies \mathbf{x}_1 = \mathbf{x}_2$
$\therefore$ The solution to square $A\mathbf{x} = \mathbf{b}$ is **unique**. QED

Solving Square Matrix Equation $AX = B$ via A^{-1}

(A is **invertible** & B, X have compatible shapes s.t. product AX is well-defined.)

A^{-1} can be used to solve square **matrix eqn** $AX = B$ for X:

$$\begin{array}{rrcll} & AX & = & B & \text{[Given Statement]} \\ \Longrightarrow & {\color{blue}A^{-1}}AX & = & {\color{blue}A^{-1}}B & \text{[Left-Multiply both sides by } A^{-1}\text{]} \\ \Longrightarrow & (A^{-1}A)X & = & A^{-1}B & \text{[Associativity of Matrix Multiplication]} \\ \Longrightarrow & (I)X & = & A^{-1}B & \text{[Definition of Inverse of } A\text{]} \\ \Longrightarrow & X & = & A^{-1}B & [I \text{ is Matrix Multiplicative Identity]} \end{array}$$

$$\therefore \quad AX = B \implies \boxed{X = A^{-1}B}$$

Solving Square Matrix Equation $XA = B$ via A^{-1}

(A is **invertible** & B, X have compatible shapes s.t. product XA is well-defined.)

A^{-1} can be used to solve square **matrix eqn** $XA = B$ for X:

$$\begin{array}{rrcll} & XA & = & B & \text{[Given Statement]} \\ \Longrightarrow & XAA^{-1} & = & BA^{-1} & \text{[Right-Multiply both sides by } A^{-1}\text{]} \\ \Longrightarrow & X(AA^{-1}) & = & BA^{-1} & \text{[Associativity of Matrix Multiplication]} \\ \Longrightarrow & X(I) & = & BA^{-1} & \text{[Definition of Inverse of } A\text{]} \\ \Longrightarrow & X & = & BA^{-1} & [I \text{ is Matrix Multiplicative Identity]} \end{array}$$

$$\therefore \quad XA = B \implies \boxed{X = BA^{-1}}$$

Fin.