Solving $A\mathbf{x} = \mathbf{b}$ w/ different $\mathbf{b}$'s: LU-Factorization

Linear Algebra

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Elementary Row Operations (Review)

Consider solving several linear systems $A\mathbf{x} = \mathbf{b}$ with same A but different $\mathbf{b}$'s:

- Performing Gauss-Jordan on each $[A|\mathbf{b}]$ repeats the same work alot!!
- Computing $\mathbf{x} = A^{-1}\mathbf{b}$ only works if A is **square AND invertible**!!

Recall the definition of an **elementary row operation**:

Definition

(Elementary Row Operations)

There are three types of **elementary row operations** applicable to $[A|\mathbf{b}]$:

(SWAP)	$[R_i \leftrightarrow R_j]$	Swap row i & row j
(SCALE)	$[\alpha R_j \rightarrow R_j]$	Multiply row j by a non-zero scalar α
(COMBINE)	$[\alpha R_i + R_j \rightarrow R_j]$	Add scalar multiple α of row i to row j

VERY IMPORTANT: For this section (LARSON 2.4) only:

- The only elementary row operation considered is **COMBINE**.
- **COMBINE** operations will be applied to A instead of $[A|\mathbf{b}]$.

COMBINE Operations (Examples)

$$\begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{bmatrix} \xrightarrow{\mathbf{5}R_1+R_2\to R_2} \begin{bmatrix} 3 & 4 & 0 \\ 14 & 20 & 1 \\ 17 & 27 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{bmatrix} \xrightarrow{\mathbf{5}R_1+R_3\to R_3} \begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 17 & 27 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{bmatrix} \xrightarrow{\mathbf{5}R_2+R_3\to R_3} \begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & 1 \\ -3 & 7 & 14 \end{bmatrix}$$

Elementary Matrices Representing SWAP Operations

It would be useful to encapsulate an elementary row op into a **matrix**:

Definition

(Elementary Matrix)

An $n \times n$ square matrix E is an **elementary matrix** if it can be obtained from the $n \times n$ identity matrix I by a single elementary row operation.

Here are some 3×3 elementary matrices:

(SWAP) $[R_1 \leftrightarrow R_2] : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $[R_2 \leftrightarrow R_3] : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

NOTE: SWAP operations will never be used in this section - the above elementary matrix representations are shown here for completion.

Elementary Matrices Representing SCALE Operations

It would be useful to encapsulate an elementary row op into a **matrix**:

Definition

(Elementary Matrix)

An $n \times n$ square matrix E is an **elementary matrix** if it can be obtained from the $n \times n$ identity matrix I by a single elementary row operation.

Here are some 3×3 elementary matrices:

(SCALE) $[3R_1 \to R_1] : \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [(-8)R_3 \to R_3] : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -8 \end{bmatrix}$

NOTE: SCALE operations will never be used in this section - the above elementary matrix representations are shown here for completion.

Elementary Matrices Representing COMBINE Op's

It would be useful to encapsulate an elementary row op into a **matrix**:

Definition

(Elementary Matrix)

An $n \times n$ square matrix E is an **elementary matrix** if it can be obtained from the $n \times n$ identity matrix I by a single elementary row operation.

Here are some 3×3 elementary matrices:

(COMBINE)

$$[5R_1 + R_2 \to R_2] : \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [(-5)R_2 + R_3 \to R_3] : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

NOTE: **All** elementary matrices considered in this section will be **COMBINE**'s.

Elementary Matrices applied to A with $m = n$

To apply an elem row op to matrix A, **left-multiply** A by elementary matrix.

$$\text{Let } A = \begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{bmatrix}. \quad \text{Then:}$$

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{bmatrix} \xrightarrow{\mathbf{5}R_1 + R_2 \to R_2} \begin{bmatrix} 3 & 4 & 0 \\ 14 & 20 & 1 \\ 2 & 7 & 9 \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{5} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{bmatrix} \xrightarrow{\mathbf{5}R_1 + R_3 \to R_3} \begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 17 & 27 & 9 \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mathbf{5} & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{bmatrix} \xrightarrow{\mathbf{5}R_2 + R_3 \to R_3} \begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & 1 \\ -3 & 7 & 14 \end{bmatrix}$$

WARNING: **Right-multiplying** A by an elementary matrix applies the corresponding elementary **column** operation.

Elementary Matrices applied to A with $m < n$

To apply an elem row op to matrix A, **left-multiply** A by elementary matrix.

$$\text{Let } A = \begin{bmatrix} 3 & 4 & 0 & 8 \\ -1 & 0 & 1 & 6 \\ 2 & 7 & 9 & 3 \end{bmatrix}. \quad \text{Then:}$$

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 & 8 \\ -1 & 0 & 1 & 6 \\ 2 & 7 & 9 & 3 \end{bmatrix} \xrightarrow{\mathbf{5}R_1 + R_2 \to R_2} \begin{bmatrix} 3 & 4 & 0 & 8 \\ 14 & 20 & 1 & 46 \\ 2 & 7 & 9 & 3 \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{5} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 & 8 \\ -1 & 0 & 1 & 6 \\ 2 & 7 & 9 & 3 \end{bmatrix} \xrightarrow{\mathbf{5}R_1 + R_3 \to R_3} \begin{bmatrix} 3 & 4 & 0 & 8 \\ -1 & 0 & 1 & 6 \\ 17 & 27 & 9 & 43 \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mathbf{5} & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 & 8 \\ -1 & 0 & 1 & 6 \\ 2 & 7 & 9 & 3 \end{bmatrix} \xrightarrow{\mathbf{5}R_2 + R_3 \to R_3} \begin{bmatrix} 3 & 4 & 0 & 8 \\ -1 & 0 & 1 & 6 \\ -3 & 7 & 14 & 33 \end{bmatrix}$$

Elementary Matrices applied to A with $m > n$

To apply an elem row op to matrix A, **left-multiply** A by elementary matrix.

$$\text{Let } A = \begin{bmatrix} 3 & 4 \\ -1 & 0 \\ 2 & 7 \end{bmatrix}. \quad \text{Then:}$$

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -1 & 0 \\ 2 & 7 \end{bmatrix} \xrightarrow{\mathbf{5}R_1+R_2 \to R_2} \begin{bmatrix} 3 & 4 \\ 14 & 20 \\ 2 & 7 \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{5} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -1 & 0 \\ 2 & 7 \end{bmatrix} \xrightarrow{\mathbf{5}R_1+R_3 \to R_3} \begin{bmatrix} 3 & 4 \\ -1 & 0 \\ 17 & 27 \end{bmatrix}$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mathbf{5} & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -1 & 0 \\ 2 & 7 \end{bmatrix} \xrightarrow{\mathbf{5}R_2+R_3 \to R_3} \begin{bmatrix} 3 & 4 \\ -1 & 0 \\ -3 & 7 \end{bmatrix}$$

Inverse of an Elementary Matrix

Theorem

(Elementary Matrices are Invertible)

*If E is an **elementary matrix**, then E^{-1} exists and is an elementary matrix.*

Question: What is the inverse of an elementary matrix?
Answer: The inverse "undoes" the elementary row operation.

ELEMENTARY MATRIX:

$$[R_1 \leftrightarrow R_2] : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[3R_1 \to R_1] : \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[5R_1 + R_2 \to R_2] : \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ITS INVERSE:

$$[R_1 \leftrightarrow R_2] : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\left(\tfrac{1}{3}\right) R_1 \to R_1\right] : \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[(-5)R_1 + R_2 \to R_2] : \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Main Diagonal of a Matrix (Definition)

Definition

(Main Diagonal of a Matrix)

The **main diagonal** of a $m \times n$ matrix comprises of all (k,k)-entries of the matrix for $k = 1, 2, \cdots, \min\{m, n\}$.

The **main diagonal** entries of each matrix below is shown in **blue**:

$$\begin{bmatrix} \mathbf{1} & 2 \\ 3 & \mathbf{4} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & 2 & 3 \\ 4 & \mathbf{5} & 6 \end{bmatrix}, \begin{bmatrix} \mathbf{1} & 2 \\ 3 & \mathbf{4} \\ 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{1} & 2 & 3 \\ 4 & \mathbf{5} & 6 \\ 7 & 8 & \mathbf{9} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & 2 & 3 & 4 \\ 5 & \mathbf{6} & 7 & 8 \\ 9 & 10 & \mathbf{11} & 12 \end{bmatrix}, \begin{bmatrix} \mathbf{1} & 2 & 3 \\ 4 & \mathbf{5} & 6 \\ 7 & 8 & \mathbf{9} \\ 10 & 11 & 12 \end{bmatrix}$$

Special Matrices: Triangular Matrices

In certain applications & higher math, **triangular matrices** are useful:

Definition

(Triangular Matrices)

Let L, U be $m \times n$ matrices. Then:

L is called **lower triangular** if entries above the main diagonal are all zero.
U is called **upper triangular** if entries below the main diagonal are all zero.

A matrix is called **triangular** if it's either lower or upper triangular.

Lower Triangular: $\begin{bmatrix} 1 & \mathbf{0} \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ 4 & 5 & \mathbf{0} \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{0} \\ 3 & 4 \\ 5 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ 4 & 5 & \mathbf{0} \\ 0 & 8 & 0 \end{bmatrix}, \cdots$

Upper Triangular: $\begin{bmatrix} 1 & 2 \\ \mathbf{0} & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ \mathbf{0} & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ \mathbf{0} & 4 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ \mathbf{0} & 0 & 6 \\ \mathbf{0} & \mathbf{0} & 9 \end{bmatrix}, \cdots$

Special Matrices: Unit Triangular Matrices

Moreover, in what's coming next, **unit triangular matrices** are useful:

Definition

(Unit Triangular Matrices)

Let L, U be $m \times n$ matrices. Then:

L is **unit lower triangular** if it's lower triangular and has all ones on main diagonal.

U is **unit upper triangular** if it's upper triangular and has all ones on main diagonal.

Unit Lower Triangular: $\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ 3 & \mathbf{1} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 4 & \mathbf{1} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ 3 & \mathbf{1} \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{1} & \mathbf{0} \\ 0 & 8 & \mathbf{1} \end{bmatrix}, \cdots$

Unit Upper Triangular: $\begin{bmatrix} \mathbf{1} & 2 \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & 0 & 3 \\ \mathbf{0} & \mathbf{1} & 1 \end{bmatrix}, \begin{bmatrix} \mathbf{1} & 2 \\ \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & 0 & 1 \\ \mathbf{0} & \mathbf{1} & 6 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \cdots$

The Value of Triangular Matrices

Consider the following 3×3 **lower triangular linear system**:

$$\left\{ \begin{array}{rcrcrcr} x_1 & & & & & = & 1 \\ 2x_1 & + & 3x_2 & & & = & -1 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 11 \end{array} \right. \iff [A|\mathbf{b}], \text{ where } A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

Then the soln can be found by a **forward-solve** (AKA **forward substitution**):

$$\begin{array}{ccccccc} x_1 = 1 & & & \implies & x_1 & = & 1 \\ 2x_1 + 3x_2 = -1 & \implies & 2(1) + 3x_2 = -1 & \implies & x_2 & = & -1 \\ 4x_1 + 5x_2 + 6x_3 = 11 & \implies & 4(1) + 5(-1) + 6x_3 = 11 & \implies & x_3 & = & 2 \end{array}$$

The cases where the lower triangular linear system has no soln or infinitely many solns can also be handled with little trouble by a forward-solve.

Having said that, for this section (LARSON 2.4), only linear systems with a **unique solution** will be considered.

The Value of Triangular Matrices

Consider the following 3×3 **upper triangular linear system**:

$$\left\{ \begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & 3 \\ & & 4x_2 & + & 5x_3 & = & 10 \\ & & & & 6x_3 & = & 12 \end{array} \right. \iff [A|\mathbf{b}], \text{ where } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Then the solution can be found by a **back-solve** (AKA **back substitution**):

$$\begin{array}{ccccccc} 6x_3 = 12 & \Longrightarrow & x_3 = 12/6 & \Longrightarrow & x_3 & = & 2 \\ 4x_2 + 5x_3 = 10 & \Longrightarrow & 4x_2 + 5(2) = 10 & \Longrightarrow & x_2 & = & 0 \\ x_1 + 2x_2 + 3x_3 = 3 & \Longrightarrow & x_1 + 2(0) + 3(2) = 3 & \Longrightarrow & x_1 & = & -3 \end{array}$$

The cases where the upper triangular linear system has no soln or infinitely many solns can also be handled with little trouble by a back-solve.

Having said that, for this section (LARSON 2.4), only linear systems with a **unique solution** will be considered.

LU-Factorization of a Matrix (Procedure)

There are times in Linear Algebra where **factoring** a matrix is quite useful. Here is the first such instance:

Proposition

(LU-Factorization of a Matrix)

GIVEN: $m \times n$ *matrix* A *where* ***no row swaps are necessary***.

TASK: *Form* $A = LU$, *(L is square unit lower triangular & U is upper triangular)*

(1) COMBINE to zero-out an entry below main diagonal: $[\alpha R_i + R_j \to R_j]$
(2) Form $m \times m$ *elementary matrix corresponding to COMBINE operation:* E
(3) Find the inverse of the elementary matrix: E^{-1}
(4) Repeat steps (1)-(3) for all such entries, top-to-bottom, left-to-right
(5) Resulting matrix is upper triangular: $U = E_k E_{k-1} \cdots E_3 E_2 E_1 A$
(6) Determine L: $E_k E_{k-1} \cdots E_2 E_1 A = U \implies A = \underbrace{E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}}_{L} U$

Products of Inverses of COMBINE Elem. Matrices

Products of inverses of COMBINE elementary matrices can be found instantly:

Proposition

(Products of Inverses of COMBINE Elementary Matrices)

Let $E_1, E_2, \cdots, E_{k-1}, E_k$ *be* $n \times n$ *COMBINE elementary matrices.*

Then the $n \times n$ *matrix product* $E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$ *is simply the unit lower triangular matrix with each entry below the main diagonal being the corresponding single non-zero entry in one of the inverse elementary matrices.*

Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -8 & 1 \end{bmatrix}$ Then:

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 8 & 1 \end{bmatrix}$$

The product $E_3 E_2 E_1$ is not obvious: $E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ \mathbf{-26} & -8 & 1 \end{bmatrix}$

Solving $A\mathbf{x} = \mathbf{b}$ via LU-Factorization of A

Proposition

(Solving $A\mathbf{x} = \mathbf{b}$ via LU-Factorization of A)

GIVEN: $m \times n$ *linear system* $A\mathbf{x} = \mathbf{b}$ *where* ***no row swaps are necessary****.*

TASK: *Solve linear system via* $A = LU$

(1) Perform LU-Factorization of A (see previous slides)
(2) Notice $A\mathbf{x} = \mathbf{b} \implies (LU)\mathbf{x} = \mathbf{b} \implies L(U\mathbf{x}) = \mathbf{b} \implies$ *Let* $\mathbf{y} = U\mathbf{x}$
(3) Solve ***square triangular system*** $L\mathbf{y} = \mathbf{b}$ *for* $\mathbf{y}$ *via forward-solve*
(4) Solve ***triangular system*** $U\mathbf{x} = \mathbf{y}$ *for* $\mathbf{x}$

- *If U is* ***square****, solve* $U\mathbf{x} = \mathbf{y}$ *via back-solve*
- *If U is non-square, solve* $U\mathbf{x} = \mathbf{y}$ *via Gauss-Jordan Elimination*

REMARK: Here, $A\mathbf{x} = \mathbf{b}$ will always be **square** & have a **unique solution**. The reason being most applications lead to square linear systems with unique solutions. Moreover, most computer algorithms can only handle square linear systems with a unique solution. Finally, using $A = LU$ is preferable to computing A^{-1} since it's too slow & unstable for a computer to invert most large square matrices. Take Numerical Linear Algebra for the details.

Solving $A\mathbf{x} = \mathbf{b}$ with different $\mathbf{b}$'s via $A = LU$

$A = LU$ is efficient when solving several linear systems with the same matrix A and different RHS $\mathbf{b}$'s since $A = LU$ only has to be computed once:

$$\begin{array}{ccccccc}
A\mathbf{x} = \mathbf{b}_1 & \Longrightarrow & L(U\mathbf{x}) = \mathbf{b}_1 & \Longrightarrow & \text{Solve } L\mathbf{y}_1 = \mathbf{b}_1 & \Longrightarrow & \text{Solve } U\mathbf{x} = \mathbf{y}_1 \\
A\mathbf{x} = \mathbf{b}_2 & \Longrightarrow & L(U\mathbf{x}) = \mathbf{b}_2 & \Longrightarrow & \text{Solve } L\mathbf{y}_2 = \mathbf{b}_2 & \Longrightarrow & \text{Solve } U\mathbf{x} = \mathbf{y}_2 \\
A\mathbf{x} = \mathbf{b}_3 & \Longrightarrow & L(U\mathbf{x}) = \mathbf{b}_3 & \Longrightarrow & \text{Solve } L\mathbf{y}_3 = \mathbf{b}_3 & \Longrightarrow & \text{Solve } U\mathbf{x} = \mathbf{y}_3 \\
A\mathbf{x} = \mathbf{b}_4 & \Longrightarrow & L(U\mathbf{x}) = \mathbf{b}_4 & \Longrightarrow & \text{Solve } L\mathbf{y}_4 = \mathbf{b}_4 & \Longrightarrow & \text{Solve } U\mathbf{x} = \mathbf{y}_4 \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

If A is **square**, computing $A = LU$ is more efficient than computing A^{-1}.

WEX 2-4-1: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(a) Find the LU-Factorization for A

$$A \xrightarrow{(-3)R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \implies E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \implies E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\implies E_1 A = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = U \implies A = E_1^{-1} U \implies L = E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\therefore \quad A = LU \iff \boxed{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}}$$

(b) Use $A = LU$ to solve linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$L\mathbf{y} = \mathbf{b} \implies \left\{ \begin{array}{rcrcr} y_1 & & & = & 1 \\ 3y_1 & + & y_2 & = & -1 \end{array} \right. \xrightarrow{\text{Forward-Solve}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$U\mathbf{x} = \mathbf{y} \implies \left\{ \begin{array}{rcrcr} x_1 & + & 2x_2 & = & 1 \\ & - & 2x_2 & = & -4 \end{array} \right. \xrightarrow{\text{Back-Solve}} \boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}}$$

Fin

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