Determinants: Intro & Cofactor Expansions

Linear Algebra

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21 September 2015

Determinant of a Square Matrix (Motivation)

Consider the prototype 2×2 square linear system $A\mathbf{x} = \mathbf{b}$:

$$\left\{ \begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array} \right. \iff A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Moreover, let $a_{11}, \cdots, a_{22}, b_1, b_2$ be chosen s.t. there's a **unique solution**.

Then $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \\ \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}} \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} b_1 a_{22} - b_2 a_{12} \\ b_2 a_{11} - b_1 a_{21} \end{bmatrix}$

Notice that the denominators of x_1 & x_2 only involve the entries of matrix A.
Moreover, notice that the denominators of x_1 & x_2 are exactly the same!

This is the case for any $n \times n$ square linear system with a unique solution.
This scalar value comes up so often in Linear Algebra that it has a name:

Definition

(Determinant of a Square Matrix - "First Principles" Definition)

Let linear system $A\mathbf{x} = \mathbf{b}$ be **square** & have a **unique** solution.
Then the denominator of the solution is called the **determinant** of matrix A.

The determinant of a non-square matrix is undefined.

Determinant of a 2×2 Square Matrix

Solving a particular linear system is alot of work & it wouldn't be obvious what the common denominator is in the solution.

There's an easier procedure to compute determinants of $n \times n$ matrices.

For 2×2 matrices, there's an extremely quick procedure:

Proposition

(Determinant of a 2×2 Square Matrix)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *where* $a, b, c, d \in \mathbb{R}$. *Then the determinant of* A *is:*

$$|A| \equiv \textit{det}(A) := ad - bc$$

ALTERNATIVE NOTATION:

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ *also represents the determinant (not the absolute value.)*

WEX 3-1-1: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = \boxed{-2}$

Minors & Cofactors of a Square Matrix

Definition

(Minor & Cofactors)

Let A be a $n \times n$ **square matrix**. Then:

- The (i,j)**-minor** of A, denoted M_{ij}, is the determinant of the matrix obtained by **removing** the i^{th} row & j^{th} column of A.
- The (i,j)**-cofactor** of A, denoted C_{ij}, is $C_{ij} := (-1)^{i+j} M_{ij}$

For instance, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & \mathbf{5} & \mathbf{6} \\ 7 & \mathbf{8} & \mathbf{9} \end{bmatrix}$, then its $(1,1)$-minor & $(1,1)$-cofactor is:

$$C_{11} = (-1)^{1+1} M_{11} = (1) \begin{vmatrix} \mathbf{5} & \mathbf{6} \\ \mathbf{8} & \mathbf{9} \end{vmatrix} = (1)\,[(5)(9) - (6)(8)] = -3$$

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For instance, if $A = \begin{bmatrix} 1 & 2 & 3 \\ \mathbf{4} & 5 & \mathbf{6} \\ \mathbf{7} & 8 & \mathbf{9} \end{bmatrix}$, then its $(1,2)$-minor & $(1,2)$-cofactor is:

$$C_{12} = (-1)^{1+2} M_{12} = (-1) \begin{vmatrix} \mathbf{4} & \mathbf{6} \\ \mathbf{7} & \mathbf{9} \end{vmatrix} = (-1)\left[(4)(9) - (6)(7)\right] = 6$$

Minors & Cofactors of a Square Matrix

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Let A be a $n \times n$ **square matrix**. Then:

- The (i,j)**-minor** of A, denoted M_{ij}, is the determinant of the matrix obtained by **removing** the i^{th} row & j^{th} column of A.
- The (i,j)**-cofactor** of A, denoted C_{ij}, is $C_{ij} := (-1)^{i+j} M_{ij}$

For instance, if $A = \begin{bmatrix} 1 & 2 & 3 \\ \mathbf{4} & \mathbf{5} & 6 \\ \mathbf{7} & \mathbf{8} & 9 \end{bmatrix}$, then its $(1,3)$-minor & $(1,3)$-cofactor is:

$$C_{13} = (-1)^{1+3} M_{13} = (1) \begin{vmatrix} \mathbf{4} & \mathbf{5} \\ \mathbf{7} & \mathbf{8} \end{vmatrix} = (1)\left[(4)(8) - (5)(7)\right] = -3$$

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Let A be a $n \times n$ **square matrix**. Then:

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For instance, if $A = \begin{bmatrix} 1 & \mathbf{2} & \mathbf{3} \\ 4 & 5 & 6 \\ 7 & \mathbf{8} & \mathbf{9} \end{bmatrix}$, then its $(2,1)$-minor & $(2,1)$-cofactor is:

$$C_{21} = (-1)^{2+1} M_{21} = (-1) \begin{vmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{8} & \mathbf{9} \end{vmatrix} = (-1)\left[(2)(9) - (3)(8)\right] = 6$$

Minors & Cofactors of a Square Matrix

Definition

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Let A be a $n \times n$ **square matrix**. Then:

- The (i,j)**-minor** of A, denoted M_{ij}, is the determinant of the matrix obtained by **removing** the i^{th} row & j^{th} column of A.
- The (i,j)**-cofactor** of A, denoted C_{ij}, is $C_{ij} := (-1)^{i+j} M_{ij}$

For instance, if $A = \begin{bmatrix} \mathbf{1} & 2 & \mathbf{3} \\ 4 & 5 & 6 \\ \mathbf{7} & 8 & \mathbf{9} \end{bmatrix}$, then its $(2,2)$-minor & $(2,2)$-cofactor is:

$$C_{22} = (-1)^{2+2} M_{22} = (1) \begin{vmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{7} & \mathbf{9} \end{vmatrix} = (1)\left[(1)(9) - (3)(7)\right] = -12$$

Minors & Cofactors of a Square Matrix

Definition

(Minor & Cofactors)

Let A be a $n \times n$ **square matrix**. Then:

- The (i,j)-**minor** of A, denoted M_{ij}, is the determinant of the matrix obtained by **removing** the i^{th} row & j^{th} column of A.
- The (i,j)-**cofactor** of A, denoted C_{ij}, is $C_{ij} := (-1)^{i+j} M_{ij}$

For instance, if $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & 3 \\ 4 & 5 & 6 \\ \mathbf{7} & \mathbf{8} & 9 \end{bmatrix}$, then its $(2,3)$-minor & $(2,3)$-cofactor is:

$$C_{23} = (-1)^{2+3} M_{23} = (-1) \begin{vmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{7} & \mathbf{8} \end{vmatrix} = (-1)\left[(1)(8) - (2)(7)\right] = 6$$

Minors & Cofactors of a Square Matrix

Definition

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Let A be a $n \times n$ **square matrix**. Then:

- The (i,j)**-minor** of A, denoted M_{ij}, is the determinant of the matrix obtained by **removing** the i^{th} row & j^{th} column of A.
- The (i,j)**-cofactor** of A, denoted C_{ij}, is $C_{ij} := (-1)^{i+j} M_{ij}$

For instance, if $A = \begin{bmatrix} 1 & \mathbf{2} & \mathbf{3} \\ 4 & \mathbf{5} & \mathbf{6} \\ 7 & 8 & 9 \end{bmatrix}$, then its $(3,1)$-minor & $(3,1)$-cofactor is:

$$C_{31} = (-1)^{3+1} M_{31} = (1) \begin{vmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{5} & \mathbf{6} \end{vmatrix} = (1)\left[(2)(6) - (3)(5)\right] = -3$$

Minors & Cofactors of a Square Matrix

Definition

(Minor & Cofactors)

Let A be a $n \times n$ **square matrix**. Then:

- The (i,j)-**minor** of A, denoted M_{ij}, is the determinant of the matrix obtained by **removing** the i^{th} row & j^{th} column of A.
- The (i,j)-**cofactor** of A, denoted C_{ij}, is $C_{ij} := (-1)^{i+j} M_{ij}$

For instance, if $A = \begin{bmatrix} \mathbf{1} & 2 & \mathbf{3} \\ \mathbf{4} & 5 & \mathbf{6} \\ 7 & 8 & 9 \end{bmatrix}$, then its $(3,2)$-minor & $(3,2)$-cofactor is:

$$C_{32} = (-1)^{3+2} M_{32} = (-1) \begin{vmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{4} & \mathbf{6} \end{vmatrix} = (-1)\left[(1)(6) - (3)(4)\right] = 6$$

Minors & Cofactors of a Square Matrix

Definition

(Minor & Cofactors)

Let A be a $n \times n$ **square matrix**. Then:

- The (i,j)**-minor** of A, denoted M_{ij}, is the determinant of the matrix obtained by **removing** the i^{th} row & j^{th} column of A.
- The (i,j)**-cofactor** of A, denoted C_{ij}, is $C_{ij} := (-1)^{i+j} M_{ij}$

For instance, if $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & 3 \\ \mathbf{4} & \mathbf{5} & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then its $(3,3)$-minor & $(3,3)$-cofactor is:

$$C_{33} = (-1)^{3+3} M_{33} = (1) \begin{vmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{4} & \mathbf{5} \end{vmatrix} = (1)\,[(1)(5) - (2)(4)] = -3$$

Determinant of $n \times n$ Matrix via Cofactor Expansion

Theorem

(Determinant via Cofactor Expansion)

*Let A be a $n \times n$ **square matrix**. Then:*

$$\textit{det}(A) = \sum_{k=1}^{n} a_{ik}C_{ik} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad (i^{th} \textit{ row expansion})$$

$$\textit{det}(A) = \sum_{k=1}^{n} a_{kj}C_{kj} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \qquad (j^{th} \textit{ column expansion})$$

WEX 3-1-2: Find the determinant of $A = \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{1} \\ 4 & 3 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

$$\begin{aligned} |A| &= (\mathbf{0})C_{11} + (\mathbf{2})C_{12} + (\mathbf{1})C_{13} \\ &= (-1)^{1+1}(0)\begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} + (-1)^{1+2}(2)\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} + (-1)^{1+3}(1)\begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} \\ &= 0 + (-2)\left[(4)(2) - (3)(1)\right] + (1)\left[(4)(1) - (3)(1)\right] \\ &= \boxed{-9} \end{aligned}$$

Determinant of $n \times n$ Matrix via Cofactor Expansion

Theorem

(Determinant via Cofactor Expansion)

*Let A be a $n \times n$ **square matrix**. Then:*

$$\text{det}(A) = \sum_{k=1}^{n} a_{ik}C_{ik} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad (i^{th} \text{ row expansion})$$

$$\text{det}(A) = \sum_{k=1}^{n} a_{kj}C_{kj} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \qquad (j^{th} \text{ column expansion})$$

WEX 3-1-2: Find the determinant of $A = \begin{bmatrix} \mathbf{0} & 2 & 1 \\ \mathbf{4} & 3 & 3 \\ \mathbf{1} & 1 & 2 \end{bmatrix}$.

$$\begin{aligned}
|A| &= (\mathbf{0})C_{11} + (\mathbf{4})C_{21} + (\mathbf{1})C_{31} \\
&= (-1)^{1+1}(0)\begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} + (-1)^{2+1}(4)\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + (-1)^{3+1}(1)\begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix} \\
&= 0 + (-4)\,[(2)(2) - (1)(1)] + (1)\,[(2)(3) - (1)(3)] \\
&= \boxed{-9}
\end{aligned}$$

Determinant of $n \times n$ Matrix via Cofactor Expansion

Theorem

(Determinant via Cofactor Expansion)

*Let A be a $n \times n$ **square matrix**. Then:*

$$\text{det}(A) = \sum_{k=1}^{n} a_{ik}C_{ik} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad (i^{th} \text{ row expansion})$$

$$\text{det}(A) = \sum_{k=1}^{n} a_{kj}C_{kj} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \qquad (j^{th} \text{ column expansion})$$

WEX 3-1-2: Find the determinant of $A = \begin{bmatrix} 0 & 2 & 1 \\ \mathbf{4} & \mathbf{3} & \mathbf{3} \\ 1 & 1 & 2 \end{bmatrix}$.

$$\begin{aligned} |A| &= (\mathbf{4})C_{21} + (\mathbf{3})C_{22} + (\mathbf{3})C_{23} \\ &= (-1)^{2+1}(4)\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + (-1)^{2+2}(3)\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + (-1)^{2+3}(3)\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} \\ &= (-4)\left[(2)(2) - (1)(1)\right] + (3)\left[(0)(2) - (1)(1)\right] + (-3)\left[(0)(1) - (2)(1)\right] \\ &= \boxed{-9} \end{aligned}$$

Determinant of $n \times n$ Matrix via Cofactor Expansion

Theorem

(Determinant via Cofactor Expansion)

*Let A be a $n \times n$ **square matrix**. Then:*

$$\det(A) = \sum_{k=1}^{n} a_{ik}C_{ik} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad (i^{th} \text{ row expansion})$$

$$\det(A) = \sum_{k=1}^{n} a_{kj}C_{kj} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \qquad (j^{th} \text{ column expansion})$$

WEX 3-1-2: Find the determinant of $A = \begin{bmatrix} 0 & 2 & \mathbf{1} \\ 4 & 3 & \mathbf{3} \\ 1 & 1 & \mathbf{2} \end{bmatrix}$.

$$\begin{aligned}
|A| &= (\mathbf{1})C_{13} + (\mathbf{3})C_{23} + (\mathbf{2})C_{33} \\
&= (-1)^{1+3}(1)\begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} + (-1)^{2+3}(3)\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + (-1)^{3+3}(2)\begin{vmatrix} 0 & 2 \\ 4 & 3 \end{vmatrix} \\
&= (1)\left[(4)(1) - (3)(1)\right] + (-3)\left[(0)(1) - (2)(1)\right] + (2)\left[(0)(3) - (2)(4)\right] \\
&= \boxed{-9}
\end{aligned}$$

Sparse & Dense Matrices

Definition

(Sparse Matrix)

A **sparse matrix** has at least several zeros.

<u>REMARK:</u> Elementary, triangular and diagonal matrices are sparse matrices.

Sparse Matrices: $\begin{bmatrix} 0 & 2 & 0 & 4 \\ 5 & 0 & 7 & 0 \\ 0 & 6 & 7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 7 & 8 \\ 3 & 6 & 7 & 0 \\ 8 & 5 & 0 & 1 \end{bmatrix}$

Definition

(Dense Matrix)

A **dense matrix** has at most a couple zeros.

Dense Matrices: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 6 & 7 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 6 & 7 & 2 \\ 8 & 5 & 0 & 1 \end{bmatrix}$

Determinant of a Sparse Matrix

Cofactor Expansions are efficient for determinants of **sparse matrices**.
It's best to expand along the row or column with the **most zeros**:

WEX 3-1-3: Find the determinant of $A = \begin{bmatrix} -2 & 0 & 5 & 0 \\ 0 & 6 & 4 & 2 \\ 0 & 0 & 0 & 8 \\ 1 & 0 & -3 & -3 \end{bmatrix}$

$$\begin{aligned}
|A| &= (0)C_{31} + (0)C_{32} + (0)C_{33} + (8)C_{34} && \text{(Cofactor Exp along row 3 of } A\text{)} \\
&= (-1)^{3+4}(8) \begin{vmatrix} -2 & 0 & 5 \\ 0 & 6 & 4 \\ 1 & 0 & -3 \end{vmatrix} \\
&= (-8) \left[0 + (6)(-1)^{2+2} \begin{vmatrix} -2 & 5 \\ 1 & -3 \end{vmatrix} \right] && \text{(Cofactor Exp along col 2 of } C_{34}\text{)} \\
&= (-8)(6)\left[(-2)(-3) - (5)(1)\right] && \text{(Det of } 2 \times 2 \text{ matrix formula)} \\
&= \boxed{-48}
\end{aligned}$$

Cofactor Expansions of $n \times n$ **dense matrices** $(n \geq 4)$ take far too much work!

A better method for dense matrices to be shown next section (LARSON 3.2)

Determinant of a Triangular Matrix

For **triangular** matrices, their determinants are straightforward to find:

Theorem

(Determinant of a Triangular Matrix)

*Let A be a $n \times n$ **triangular** matrix. Then, det$(A) = a_{11}a_{22}a_{33}\cdots a_{nn}$*

i.e. determinant of a triangular matrix is the product of the diagonal entries.

WEX 3-1-4: Find the determinant of $A = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & -3 & -4 & 4 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

det$(A) = a_{11}a_{22}a_{33}a_{44} = (1)(-3)(5)(-1) = \boxed{15}$

Fin

Fin.