Determinants: Elementary Row/Column Operations

Linear Algebra

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Elementary Row Operations & Determinants

Theorem

(Elementary Row Operations & Determinants)
*Recall the **elementary row operations**:*

(SWAP)	$[R_i \leftrightarrow R_j]$	*Swap row i & row j*
(SCALE)	$[\alpha R_j \rightarrow R_j]$	*Multiply row j by a non-zero scalar α*
(COMBINE)	$[\alpha R_i + R_j \rightarrow R_j]$	*Add scalar multiple α of row i to row j*

Let A, B be $n \times n$ square matrices. Then:

$$(\textit{ROW SWAP}) \quad \textit{If} \quad A \xrightarrow{R_i \leftrightarrow R_j} B, \textit{ then } \det(B) = -\det(A)$$

$$(\textit{ROW SCALE}) \quad \textit{If} \quad A \xrightarrow{\alpha R_j \rightarrow R_j} B, \textit{ then } \det(B) = \alpha \det(A)$$

$$(\textit{ROW COMBINE}) \quad \textit{If} \quad A \xrightarrow{\alpha R_i + R_j \rightarrow R_j} B, \textit{ then } \det(B) = \det(A)$$

*i.e. Performing a **row swap** causes the determinant to **change sign**.*
*i.e. Performing a **row scale by** α causes the determinant to **multiplied by** α.*
*i.e. Performing a **row combine** causes the determinant to **remain the same**.*

PROOF: Requires **proof-by-induction** - see the textbook if interested.

Elementary Column Operations & Determinants

Theorem

(Elementary Column Operations & Determinants)
*Recall the **elementary column operations**:*

(SWAP)	$[C_i \leftrightarrow C_j]$	*Swap column i & row j*
(SCALE)	$[\alpha C_j \to C_j]$	*Multiply column j by a non-zero scalar α*
(COMBINE)	$[\alpha C_i + C_j \to C_j]$	*Add scalar multiple α of column i to column j*

Let A, B be $n \times n$ square matrices. Then:

(*COLUMN SWAP*)	*If*	$A \xrightarrow{C_i \leftrightarrow C_j} B$	*, then* $\det(B) = -\det(A)$
(*COLUMN SCALE*)	*If*	$A \xrightarrow{\alpha C_j \to C_j} B$	*, then* $\det(B) = \alpha \det(A)$
(*COLUMN COMBINE*)	*If*	$A \xrightarrow{\alpha C_i + C_j \to C_j} B$	*, then* $\det(B) = \det(A)$

*i.e. Performing a **column swap** causes the determinant to **change sign**.*
*i.e. Performing a **column scale by** α causes determinant to **multiplied by** α.*
*i.e. Performing a **column combine** causes determinant to **remain the same**.*

PROOF: Requires **proof-by-induction** - see the textbook if interested.

Elem. Row Operations & Determinants (Examples)

(ROW SWAP) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \xrightarrow{R_2 \leftrightarrow R_3} (-1)\begin{vmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{vmatrix}$

(ROW SCALE) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \xrightarrow{(4)R_2 \to R_2} \left(\frac{1}{4}\right)\begin{vmatrix} 1 & 2 & 3 \\ 16 & 20 & 24 \\ 7 & 8 & 9 \end{vmatrix}$

(ROW COMBINE) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \xrightarrow{(-3)R_1 + R_3 \to R_3} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 2 & 0 \end{vmatrix}$

Elem. Col Operations & Determinants (Examples)

(COLUMN SWAP) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \xrightarrow{C_2 \leftrightarrow C_3} (-1)\begin{vmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{vmatrix}$

(COLUMN SCALE) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \xrightarrow{(4)C_2 \to C_2} \left(\frac{1}{4}\right)\begin{vmatrix} 1 & 8 & 3 \\ 4 & 20 & 6 \\ 7 & 32 & 9 \end{vmatrix}$

(COLUMN COMBINE) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \xrightarrow{(-3)C_1 + C_3 \to C_3} \begin{vmatrix} 1 & 2 & 0 \\ 4 & 5 & -6 \\ 7 & 8 & -12 \end{vmatrix}$

Row-Equivalent Matrices & Determinants

Definition

(Row-Equivalent Matrices)
Let A, B be $m \times n$ matrices.
Then A and B are **row-equivalent** if B can be obtained from A by elementary row operations.

Since performing elementary row operations causes the determinant to be multiplied by a scalar, it follows that the determinants of two row-equivalent matrices differ by a scalar multiple:

Corollary

(Determinants of Row-Equivalent Matrices)
Let A, B be $n \times n$ square matrices.
Then if A and B are row-equivalent, then det$(B) = \beta$det(A) for some $\beta \neq 0$.

Column-Equivalent Matrices & Determinants

Definition

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Let A, B be $m \times n$ matrices.
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Corollary

(Determinants of Column-Equivalent Matrices)
Let A, B be $n \times n$ square matrices.
Then if A and B are column-equivalent, then $\det(B) = \beta \det(A)$ for some $\beta \neq 0$.

Matrices with a Zero Determinant

Theorem

(Conditions that yield a Zero Determinant)

Let A be a $n \times n$ square matrix. Then:

(Z1) If an entire row of A consists of all zeros, then det$(A) = 0$
(Z2) If an entire column of A consists of all zeros, then det$(A) = 0$
(Z3) If two rows of A are equal, then det$(A) = 0$
(Z4) If two columns of A are equal, then det$(A) = 0$
(Z5) If one row of A is a multiple of another row of A, then det$(A) = 0$
(Z6) If one column of A is a multiple of another column of A, then det$(A) = 0$

$$\begin{vmatrix} 1 & 2 & 3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 7 & 8 & 9 \end{vmatrix} \overset{Z1}{=} 0, \quad \begin{vmatrix} 1 & 2 & \mathbf{0} \\ 4 & 5 & \mathbf{0} \\ 7 & 8 & \mathbf{0} \end{vmatrix} \overset{Z2}{=} 0, \quad \begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 7 & 8 & 9 \end{vmatrix} \overset{Z3}{=} 0, \quad \begin{vmatrix} 1 & \mathbf{2} & \mathbf{2} \\ 4 & \mathbf{5} & \mathbf{5} \\ 7 & \mathbf{8} & \mathbf{8} \end{vmatrix} \overset{Z4}{=} 0$$

$$\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} & \mathbf{6} \\ 7 & 8 & 9 \end{vmatrix} \overset{Z5}{=} 0, \quad \begin{vmatrix} 1 & \mathbf{2} & \mathbf{10} \\ 4 & \mathbf{5} & \mathbf{25} \\ 7 & \mathbf{8} & \mathbf{40} \end{vmatrix} \overset{Z6}{=} 0$$

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(Z6) If one column of A is a multiple of another column of A, then det$(A) = 0$

PROOF: (WLOG $\equiv$ "**W**ithout **L**oss **O**f **G**enerality")

(Z1): WLOG, let the k^{th} row of A consist of all zeros. Then:

$$\begin{aligned} |A| &= a_{k1}C_{k1} + a_{k2}C_{k2} + \cdots + a_{kn}C_{kn} && \text{[Cofactor Expansion along } k^{th} \text{ row of } A] \\ &= (0)C_{k1} + (0)C_{k2} + \cdots + (0)C_{kn} && \text{[Every entry in the } k^{th} \text{ row is zero]} \\ &= 0 \end{aligned}$$

QED

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PROOF: (WLOG $\equiv$ "**W**ithout **L**oss **O**f **G**enerality")

(Z2): WLOG, let the k^{th} column of A consist of all zeros. Then:

$$\begin{aligned} |A| &= a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} && [\text{Cofactor Expansion along } k^{th} \text{ column}] \\ &= (0)C_{1k} + (0)C_{2k} + \cdots + (0)C_{nk} && [\text{Every entry in the } k^{th} \text{ column is zero}] \\ &= 0 \end{aligned}$$

QED

Matrices with a Zero Determinant

Theorem

(Conditions that yield a Zero Determinant)

Let A be a $n \times n$ square matrix. Then:

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PROOF: (WLOG $\equiv$ "**W**ithout **L**oss **O**f **G**enerality")

(Z4): WLOG, let columns i & j of A be equal.

Then $A \xrightarrow{(-1)C_i + C_j \to C_j} B$

$\Longrightarrow$ column j of B consists of all zeros $\overset{Z2}{\Longrightarrow}$ det$(B) = 0$.

Since A, B are column-equivalent, det$(A) = \beta$det$(B) = (\beta)(0) = 0$. QED

Matrices with a Zero Determinant

Theorem

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Let A be a $n \times n$ square matrix. Then:

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PROOF: (WLOG $\equiv$ "**W**ithout **L**oss **O**f **G**enerality")

(Z5): WLOG, let $\alpha \neq 0$ and (row j of A) $= \alpha \times$ (row i of A).

Then $A \xrightarrow{(-\alpha)R_i + R_j \to R_j} B$

$\implies$ row j of B consists of all zeros $\overset{Z1}{\implies}$ det$(B) = 0$.

Since A, B are row-equivalent, det$(A) = \beta$det$(B) = (\beta)(0) = 0$. QED

Finding Determinants via Elem Row/Col Operations

Since the determinant of a **triangular** matrix is simply the product of its main diagonal entries, it's best to row/column-reduce a dense matrix down to a triangular matrix.

If in the process the matrix reduces to a matrix with a row or column consisting of all zeros, then the determinant is zero:

Proposition

(Finding Determinants via Elementary Row/Column Operations)

GIVEN: $n \times n$ *square* ***dense matrix*** A

TASK: *Find the determinant of* A

(1) Perform elem. row or column op's until one of the following is attained:

- *A matrix with a row or column of all zeros (whose determinant is zero)*
- *A triangular matrix (whose determinant is product of main diagonal)*

($\star$) Keep track of each scalar factor resulting from a SWAP or SCALE row op.
($\star$) It's best to use only ***row*** *ops – column ops will never be used again.*

Finding Determinants via Elem Row/Col Operations

WEX 3-2-1: Find the determinant of $A = \begin{bmatrix} 2 & 4 & 8 \\ 3 & 5 & 5 \\ 10 & 20 & 50 \end{bmatrix}$.

$$\begin{vmatrix} 2 & 4 & 8 \\ 3 & 5 & 5 \\ 10 & 20 & 50 \end{vmatrix} \xrightarrow{(-5)R_1+R_3 \to R_3} \begin{vmatrix} 2 & 4 & 8 \\ 3 & 5 & 5 \\ 0 & 0 & 10 \end{vmatrix} \xrightarrow{\left(\frac{1}{2}\right)R_1 \to R_1} (2)\begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 5 \\ 0 & 0 & 10 \end{vmatrix}$$

$$\xrightarrow{(-3)R_1+R_2 \to R_2} (2)\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & 10 \end{vmatrix} = (2)[(1)(-1)(10)] = \boxed{-20}$$

WEX 3-2-2: Find the determinant of $A = \begin{bmatrix} 2 & 4 & 8 \\ 3 & 3 & 12 \\ 10 & 20 & 40 \end{bmatrix}$.

$$\begin{vmatrix} 2 & 4 & 8 \\ 3 & 3 & 12 \\ 10 & 20 & 40 \end{vmatrix} \xrightarrow{(-4)C_1+C_3 \to C_3} \begin{vmatrix} 2 & 4 & 0 \\ 3 & 3 & 0 \\ 10 & 20 & 0 \end{vmatrix} \overset{Z2}{=} \boxed{0}$$

Cofactor Expansions vs. Elem Row/Col Operations

For large $n \times n$ square **dense matrices** $(n \geq 4)$, elementary row operations require far less work than a cofactor expansion:

	COFACTOR EXPANSION		**ELEM. ROW OPERATIONS**	
n	ADDITIONS	MULTIPLICATIONS	ADDITIONS	MULTIPLICATIONS
2	1	2	1	3
3	5	9	5	10
4	23	40	14	23
5	119	205	30	44
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	**3,628,799**	**6,235,300**	285	339

Even for computers, finding 10×10 determinants are significantly faster using elementary row operations!

Fin.