Determinants: Products, Inverses, Transposes

Linear Algebra

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TTU

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Determinants of Products & Transposes

Determinants of products & transposes of matrices can easily be found once the determinants of the matrices themselves are known:

Theorem

(Determinants of Products & Transposes)

Let A, B *be* $n \times n$ *square matrices and* $\alpha \neq 0$. *Then:*

(*D1*)	$\lvert AB\rvert = \lvert A\rvert\lvert B\rvert$	*Determinant of a Matrix Product*
(*D2*)	$\lvert \alpha A\rvert = \alpha^n \lvert A\rvert$	*Determinant of a Scalar Product*
(*D3*)	$\lvert A^T\rvert = \lvert A\rvert$	*Determinant of a Transpose*

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PROOF:

(D1) The proof's a bit long & tedious - see the textbook if interested.

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PROOF:

(D2) Observe that αA effectively scales each row of A by α

$$\implies A \xrightarrow{n \text{ row SCALE's by } \alpha} \alpha A \implies |\alpha A| = \underbrace{\alpha\alpha\cdots\alpha}_{n \text{ factors}} |A| = \alpha^n |A| \qquad \text{QED}$$

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$(D3)$	$\|A^T\| = \|A\|$	*Determinant of a Transpose*

<u>PROOF:</u>

(D3) Requires **proof-by-induction** - see textbook if interested.

Value of Determinant of Scalar Product Property (D2)

(D2) is useful when computing determinants of matrices with...

- ...lots of negative signs: $\begin{vmatrix} -1 & -2 & 3 \\ 4 & -5 & -6 \\ -7 & 8 & -9 \end{vmatrix} = (-1)^3 \begin{vmatrix} 1 & 2 & -3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix}$
- ...lots of large numbers: $\begin{vmatrix} -240 & 360 & 0 \\ 600 & 120 & -120 \\ 0 & 960 & 720 \end{vmatrix} = (120)^3 \begin{vmatrix} -2 & 3 & 0 \\ 5 & 1 & -1 \\ 0 & 8 & 6 \end{vmatrix}$
- ...lots of fractions: $\begin{vmatrix} 1/4 & -1/3 & 1/2 \\ -5/6 & 2/3 & 0 \\ 3/2 & 0 & -3/4 \end{vmatrix} = \left(\frac{1}{12}\right)^3 \begin{vmatrix} 3 & -4 & 6 \\ -10 & 8 & 0 \\ 18 & 0 & -9 \end{vmatrix}$

The resulting determinants are far easier to compute.

Determinants of Extended Matrix Products & Powers

Corollary

(Determinants of Extended Matrix Products)

Let $A_1, A_2, \ldots, A_k$ *be* $n \times n$ *matrices. Then:*

$(D4)$ $|A_1 A_2 \cdots A_k| = |A_1||A_2| \cdots |A_k|$ *Determinant of an Extended Product*

PROOF: Use associativity of matrix products & (D1) repeatedly. QED

It turns out the determinant of a **power** of a square matrix can be computed without actually computing the power:

Corollary

(Determinants of Powers of a Square Matrix)

Let A *be a* $n \times n$ *square matrix and* $k \geq 2$ *be a* ***positive integer****. Then:*

$(D5)$ $|A^k| = |A|^k$ *Determinant of a Power*

PROOF: $$|A^k| = |\underbrace{AA\cdots A}_{k \text{ factors}}| \stackrel{D4}{=} \underbrace{|A||A|\cdots|A|}_{k \text{ factors}} = |A|^k$$

The Determinant "Determines" Invertibility (of a Matrix)

Theorem

(Determinant "Determines" Invertibility of a Matrix)

A square matrix A *is invertible* $\iff |A| \neq 0$

PROOF:

$(\Rightarrow)$: Suppose A is invertible. Then:

$$
\begin{array}{llll}
 & A^{-1}A = I & & [\text{Definition of } A \text{ being invertible}] \\
\implies & |A^{-1}A| = |I| & & [\text{Take determinant of both sides of matrix eqn}] \\
\implies & |A^{-1}||A| = |I| & & [\text{Determinant of matrix product property (D1)}] \\
\implies & |A^{-1}||A| = 1 & & [\text{Determinant of identity matrix } I \text{ is one}] \\
\implies & |A^{-1}| \neq 0 \text{ and } |A| \neq 0 & & [ab \neq 0 \implies a \neq 0 \text{ and } b \neq 0]
\end{array}
$$

$(\Leftarrow)$: Suppose $|A| \neq 0$. Then $A \xrightarrow{Gauss-Jordan} I$ (since $|I| \neq 0$)

$\implies [A|I] \xrightarrow{Gauss-Jordan} [I|A^{-1}] \implies A^{-1}$ exists $\implies A$ is invertible. QED

Determinant of an Inverse

The determinant of an **inverse** can be found without finding the inverse:

Theorem

(Determinant of an Inverse)

Let A be a $n \times n$ ***invertible*** *square matrix. Then:*

$(D6)$ $\quad |A^{-1}| = \frac{1}{|A|}$ *Determinant of an Inverse*

<u>PROOF:</u> Let A be invertible. Then $|A| \neq 0$ and :

$$
\begin{array}{lll}
 & A^{-1}A = I & \text{[Definition of } A \text{ being invertible]} \\
\Longrightarrow & |A^{-1}A| = |I| & \text{[Take determinant of both sides of matrix eqn]} \\
\Longrightarrow & |A^{-1}||A| = |I| & \text{[Determinant of matrix product property (D1)]} \\
\Longrightarrow & |A^{-1}||A| = 1 & \text{[Determinant of identity matrix } I \text{ is one]} \\
\Longrightarrow & |A^{-1}| = \dfrac{1}{|A|} & \text{[Divide both sides by } |A| \text{ which is safe since } |A| \neq 0]
\end{array}
$$

QED

Determinants of Sums or Differences (WARNING)

In general, $|A+B| \neq |A|+|B|$ and $|A-B| \neq |A|-|B|$:

Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 \\ 4 & 4 \end{bmatrix} \implies |A| = -2$ and $|B| = 32$

Then $A+B = \begin{bmatrix} 4 & -3 \\ 7 & 8 \end{bmatrix}$ and $A-B = \begin{bmatrix} -2 & 7 \\ -1 & 0 \end{bmatrix}$

$\implies |A+B| = 53 \neq -2+32 = |A|+|B|$

$\implies |A-B| = 7 \neq -2-32 = |A|-|B|$

Equivalent Conditions for an Invertible Matrix

Much of what's been covered so far in the course can be summarized as so:

Proposition

(Equivalent Conditions for an Invertible Matrix)

Let A be a $n \times n$ square matrix. Then the following are equivalent:

- A *is invertible*
- $A\mathbf{x} = \mathbf{b}$ *has a* ***unique*** *soln for every RHS column vector* $\mathbf{b}$
- $A\mathbf{x} = \vec{\mathbf{0}}$ *has only the* ***trivial*** *soln* $\mathbf{x} = \vec{\mathbf{0}}$ *(i.e.* $x_1 = 0, x_2 = 0, \ldots, x_n = 0$*)*
- A *is row-equivalent to the identity matrix* I
- A *can be written as a product of* ***elementary*** *matrices*
- $|A| \neq 0$

NOTATION: $\vec{\mathbf{0}}$ denotes the **column vector** with all entries being **zero**.

Fin.