Adjoints, Cramer's Rule, Geometric Applications

Linear Algebra

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TTU

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Adjoint of a Matrix (Definition)

The **adjoint** of a square matrix is useful in some parts of higher math:

Definition

(Adjoint of a Matrix)

The **adjoint** of $n \times n$ square matrix A is defined as:

$$\text{adj}(A) := \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

i.e. The adjoint of A is the <u>tranpose</u> of the matrix of <u>cofactors</u> of A.

Finding an Inverse of a Matrix via its Adjoint

The adjoint provides yet another way to find the inverse of a matrix:

Theorem

(The Inverse of a Matrix in terms of its Adjoint)

*If A is $n \times n$ **invertible**, then $A^{-1} = \frac{1}{\text{det}(A)}\text{adj}(A)$.*

*IMPORTANT: Since all **cofactors** of A are necessary to find adj(A), use a **cofactor expansion** to quickly find det(A) afterwards.*

PROOF: The proof's subtle - see the textbook if interested.

Adjoint of a Matrix (Example)

WEX 3-4-1: Let $A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & 4 \\ 0 & 6 & 8 \end{bmatrix}$. Find adj$(A)$ and A^{-1}.

First, find all **cofactors** of matrix A:

$$\begin{aligned} C_{11} &= 0 & C_{12} &= 8 & C_{13} &= -6 \\ C_{21} &= -4 & C_{22} &= 8 & C_{23} &= -6 \\ C_{31} &= 2 & C_{32} &= -6 & C_{33} &= 5 \end{aligned}$$

Then, adj$(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -4 & 2 \\ 8 & 8 & -6 \\ -6 & -6 & 5 \end{bmatrix}}$

and det$(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = (1)(0) + (-1)(-4) + (0)(2) = 4$

$$\implies A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \begin{bmatrix} 0 & -1 & 1/2 \\ 2 & 2 & -3/2 \\ -3/2 & -3/2 & 5/4 \end{bmatrix}$$

The Value of Finding Inverses via Adjoints

Finding the inverse of a 3×3 or larger matrix via its adjoint is most useful when the entries of the matrix are **scalar functions** instead of just scalars:

$$\begin{bmatrix} e^{-t} & e^{2t} & 6e^{3t} \\ -e^{t} & 2e^{-t} & 5e^{4t} \\ -e^{6t} & -3e^{-3t} & 8e^{t} \end{bmatrix}, \quad \begin{bmatrix} (2-x-x^3) & (8+3x^2+4x^3) & (x-x^3) \\ (1-x-x^2) & (7-7x^2) & (x^2-x^3) \\ (4x-3x^2) & (9-x^3) & (1-x^3) \end{bmatrix}, \ldots.$$

Attempting to augment such matrices with the identity matrix and performing Gauss-Jordan Elimination would be extremely tedious & messy!!

Square System Solns as Determinants (Motivation)

Consider the prototype 2×2 **square** linear system $A\mathbf{x} = \mathbf{b}$:

$$\left\{ \begin{array}{lclcl} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array} \right. \iff A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Moreover, let $a_{11}, \cdots, a_{22}, b_1, b_2$ be chosen s.t. there's a **unique solution**.

Then $x_1 = \dfrac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}}$ and $x_2 = \dfrac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$

The numerators & denominators of the soln can be written as determinants:

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \quad \text{and} \quad x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

where matrix A_1 is A but with $\mathbf{b}$ as its 1^{st} column.
and matrix A_2 is A but with $\mathbf{b}$ as its 2^{nd} column.

This generalizes to $n \times n$ linear systems and is called **Cramer's Rule**.

Cramer's Rule

Theorem

(Cramer's Rule)

*Given a square $n \times n$ linear system $A\mathbf{x} = \mathbf{b}$ with a **unique** solution. Then:*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \cdots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where the k^{th} column of A_k is the column vector $\mathbf{b}$.

PROOF: See the textbook if interested.

WEX 3-4-2: Solve using Cramer's Rule: $\begin{cases} x_1 + x_2 = 2 \\ x_1 - x_2 = 3 \end{cases}$

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-5}{-2} = \frac{5}{2}, \quad x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{1}{-2} = -\frac{1}{2}$$

$$\therefore \quad (x_1, x_2) = \boxed{\left(\tfrac{5}{2}, -\tfrac{1}{2}\right)}$$

The Value of Cramer's Rule

As just observed, solving a square $n \times n$ linear system using Cramer's Rule would require computing $(n+1)$ determinants, which for $n \geq 4$ would be quite a bit of work!!

Hence, Cramer's Rule is never used to solve linear systems in practice.

Even computers don't use Cramer's Rule for solving large linear systems because there are much faster algorithms available.

The true value of Cramer's Rule lies in its use in certain **proofs**.

Areas of Triangles via Determinants

Determinants are useful in **analytic geometry** w.r.t. areas of triangles:

Proposition

(Area of a Triangle in the xy-plane)

The area of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is

$$\textit{Area} = \pm\frac{1}{2}\textit{det}\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

*where the sign $(\pm)$ is chosen to ensure the area is **positive**.*

Using determinants sidesteps to need to deal with cross products of vectors as seen in Calculus.

Volumes of Tetrahedra via Determinants

Determinants are useful in **analytic geometry** w.r.t. volumes of tetrahedra:

Proposition

(Volume of a Tetrahedron in xyz-space)

The volume of a tetrahedron with vertices $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$ is

$$\textit{Volume} = \pm\frac{1}{6}\textit{det}\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix}$$

where the sign $(\pm)$ is chosen to ensure the volume is ***positive****.*

Using determinants sidesteps to need to deal with cross products of vectors as seen in Calculus.

Collinearity/Coplanarity of Points via Determinants

Determinants are useful in **analytic geometry** w.r.t. collinearity/coplanarity:

Proposition

(Test for Collinearity of 3 Points in the xy-plane)

Let points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and $P_3 = (x_3, y_3)$.
Then:

Points P_1, P_2, P_3 are ***collinear*** $\iff$ *det* $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$

Proposition

(Test for Coplanarity of 4 Points in xyz-space)

Let points $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$, $P_3 = (x_3, y_3, z_3)$ and $P_4 = (x_4, y_4, z_4)$.
Then:

Points P_1, P_2, P_3, P_4 are ***coplanar*** $\iff$ *det* $\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} = 0$

Fin.