Vector Spaces

Linear Algebra

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Tuple Form: An Alternative Notation for Vectors

When working with vectors, writing them as column vectors may take up too much vertical space.

Instead, using the **tuple form** is an alternative notation that conserves vertical space:

$$\mathbf{u} = (1, 2, 3, 4) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \mathbf{v} = (-1, 0, 5, -3) = \begin{bmatrix} -1 \\ 0 \\ 5 \\ -3 \end{bmatrix}$$

Of course, vector addition & scalar multiplication behave exactly the same:

$$\implies \mathbf{u} + \mathbf{v} = (1, 2, 3, 4) + (-1, 0, 5, -3) = (0, 2, 8, 1)$$

$$\implies 7\mathbf{u} = (7, 14, 21, 28) \quad \text{and} \quad -3\mathbf{v} = (3, 0, -15, 9)$$

Here's what the **zero vector** looks like in this case: $\vec{\mathbf{0}} = (0, 0, 0, 0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Vector Space (Definition)

Definition

(Vector Space)

Let V be a set on which vector addition and scalar multiplication are defined. Then V is a **vector space** if these axioms hold $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall \alpha, \beta \in \mathbb{R}$:

$\mathbf{u} + \mathbf{v} \in V$	(Closure under Addition)
$\alpha \mathbf{v} \in V$	(Closure under Scalar Multiplication)
$\vec{\mathbf{0}} \in V$	(Containment of the Zero Vector)
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	(Commutativity of Vector Addition)
$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	(Associativity of Vector Addition)
$\mathbf{v} + \vec{\mathbf{0}} = \mathbf{v}$	(Additive Identity = Zero Vector)
$\mathbf{v} + (-\mathbf{v}) = \vec{\mathbf{0}}$	(Vector + Its Additive Inverse = Additive Identity)
$\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$	(Scalar Mult. Distributes over Vector Addition)
$(\alpha + \beta)\mathbf{v} = \alpha \mathbf{v} + \beta \mathbf{v}$	(Scalar Mult. Distributes over Scalar Addition)
$\alpha(\beta \mathbf{v}) = (\alpha\beta)\mathbf{v}$	(Associativity of Scalar Multiplication)
$1(\mathbf{v}) = \mathbf{v}$	(Scalar Multiplicative Identity)

Common Vector Spaces

$\mathbb{R}$	$\equiv$	Set of all real numbers (scalars)
$\mathbb{R}^2$	$\equiv$	Set of all ordered pairs (2-wide vectors)
$\mathbb{R}^3$	$\equiv$	Set of all ordered triples (3-wide vectors)
$\mathbb{R}^n$	$\equiv$	Set of all ordered n-tuples $($$n$-wide vectors$)$
$\mathbb{R}^{m\times n}$	$\equiv$	Set of all $m \times n$ matrices
$\mathbb{R}^{n\times n}$	$\equiv$	Set of all $n \times n$ square matrices
P	$\equiv$	Set of all polynomials
P_n	$\equiv$	Set of all polynomials of degree n or less
$C[a,b]$	$\equiv$	Set of all continuous functions on $[a,b]$
$C^1[a,b]$	$\equiv$	Set of all differentiable functions on $[a,b]$
$C^2[a,b]$	$\equiv$	Set of all twice-differentiable fcns on $[a,b]$
$C(-\infty,\infty)$	$\equiv$	Set of all continuous functions on $(-\infty,\infty)$

REMARK: Always assume that the operations of vector addition & scalar multiplication are the standard definitions.

One could define these operations in other ways, but such scenarios are dealt with extensively in **Abstract Algebra**. (MATH 3360)

Common Vector Spaces

VECTOR SPACE	EXAMPLE "VECTORS"	"ZERO VECTOR"
$\mathbb{R}$	Scalars: $a = -3/2, b = \sqrt{2}, c = \pi$	0
$\mathbb{R}^2$ $\mathbb{R}^3$	Vectors: $\mathbf{u} = (-3, 4), \mathbf{v} = (\sqrt{2}, \pi)$ Vectors: $(1, 1, 1), (\sqrt{2}, \pi, -1)$	$\vec{\mathbf{0}} = (0, 0)$ $\vec{\mathbf{0}} = (0, 0, 0)$
$\mathbb{R}^{3\times 2}$	3×2 Matrices: $A = \begin{bmatrix} 1 & 2 \\ -3 & \sqrt{5} \\ -\pi & 1/6 \end{bmatrix}$	$O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\mathbb{R}^{2\times 2}$	2×2 Matrices: $B = \begin{bmatrix} 1 & 2 \\ -3 & \sqrt{5} \end{bmatrix}$	$O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
P_1 P_2 P_3	Polynomials: $p(t) = 3, q(t) = 4 - 2t$ Polynomials: $3, 4 - 2t, 5 + t - 7t^2$ Polynomials: $3 - 4t + 2t^2 + 5t^3$	$z(t) = 0 + 0t$ $z(t) = 0 + 0t + 0t^2$ $0 + 0t + 0t^2 + 0t^3$
$C[0, 1]$ $C(-\infty, \infty)$ $C^1(-\infty, \infty)$	Functions: $x^2, \sin x, \sqrt{1 + x}, \frac{1}{x-2}$ Functions: $x^2, \sin x, e^x, \sqrt[3]{x}, \lvert x \rvert$ Functions: $f(x) = x^2, \sin x, e^x$	$z(x) = 0$ on $[0, 1]$ $z(x) = 0$ $z(x) = 0$

REMARK: Always assume that the operations of vector addition & scalar multiplication are the standard definitions.

Establishing that a Set is a Vector Space (Example)

WEX 4-2-1: Show that $S := \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$ is a vector space.

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First, realize that for $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to be in set S, $x_2 = 0$.

i.e., the 2^{nd} component of a vector must be zero.

Establishing that a Set is a Vector Space (Example)

<u>WEX 4-2-1:</u> Show that $S := \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$ is a vector space.

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \in S$, $\mathbf{v} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} \in S$, $\mathbf{w} = \begin{bmatrix} w_1 \\ 0 \end{bmatrix} \in S$, and $\alpha, \beta \in \mathbb{R}$. Then:

Closure of Vector Addition: $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} + \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ 0 \end{bmatrix} \in S$

Closure of Scalar Multiplication: $\alpha \mathbf{v} = \alpha \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha v_1 \\ 0 \end{bmatrix} \in S$

Containment of Zero Vector: $\vec{\mathbf{0}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S$

Establishing that a Set is a Vector Space (Example)

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Let $\mathbf{u} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \in S$, $\mathbf{v} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} \in S$, $\mathbf{w} = \begin{bmatrix} w_1 \\ 0 \end{bmatrix} \in S$, and $\alpha, \beta \in \mathbb{R}$. Then:

Commutativity of Vector Addition: $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ 0 \end{bmatrix} = \mathbf{v} + \mathbf{u}$

Associativity of Vector Addition:

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = \begin{bmatrix} u_1 + (v_1 + w_1) \\ 0 \end{bmatrix} = \begin{bmatrix} (u_1 + v_1) + w_1 \\ 0 \end{bmatrix} = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

Additive Identity: $\mathbf{v} + \vec{\mathbf{0}} = \begin{bmatrix} v_1 + 0 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \mathbf{v}$

Additive Inverse: $\mathbf{v} + (-\mathbf{v}) = \begin{bmatrix} v_1 + (-v_1) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{\mathbf{0}}$

Establishing that a Set is a Vector Space (Example)

WEX 4-2-1: Show that $S := \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$ is a vector space.

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \in S$, $\mathbf{v} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} \in S$, $\mathbf{w} = \begin{bmatrix} w_1 \\ 0 \end{bmatrix} \in S$, and $\alpha, \beta \in \mathbb{R}$. Then:

Distribution of SM over VA: $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \begin{bmatrix} u_1 + v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha(v_1 + u_1) \\ 0 \end{bmatrix} =$
$\begin{bmatrix} \alpha v_1 + \alpha u_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha v_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha u_1 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} v_1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} u_1 \\ 0 \end{bmatrix} = \alpha \mathbf{v} + \alpha \mathbf{u}$

Distribution of SM over SA: $(\alpha + \beta)\mathbf{v} = (\alpha + \beta) \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} (\alpha + \beta) v_1 \\ 0 \end{bmatrix} =$
$\begin{bmatrix} \alpha v_1 + \beta v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha v_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \beta v_1 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} v_1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \alpha \mathbf{v} + \beta \mathbf{v}$

Associativity of SM: $\alpha\left(\beta \mathbf{v}\right) = \alpha \left(\beta \begin{bmatrix} v_1 \\ 0 \end{bmatrix} \right) = \alpha \begin{bmatrix} \beta v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \beta v_1 \\ 0 \end{bmatrix} =$
$\begin{bmatrix} (\alpha\beta) v_1 \\ 0 \end{bmatrix} = (\alpha\beta) \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = (\alpha\beta)\mathbf{v}$

Scalar Multiplicative Identity: $1(\mathbf{v}) = 1 \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(v_1) \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \mathbf{v}$

Establishing that a Set is **not** a Vector Space

As just witnessed, showing that a set is a vector space is long & tedious.

Fortunately, showing that a set is **not** a vector space is far less tedious:

Corollary

(When a Set is not a Vector Space)

*A set S is **not** a vector space if at least one of the following is true:*

- *The zero vector is not in the set:* $\vec{\mathbf{0}} \notin S$
- *The additive inverse is not in the set:* $\mathbf{v} \in S$ *and* $-\mathbf{v} \notin S$
- *Closure of Vector Addition fails:* $\exists \mathbf{u}, \mathbf{v} \in S$ *such that* $\mathbf{u} + \mathbf{v} \notin S$
- *Closure of Scalar Multiplication fails:* $\exists \mathbf{v} \in S, \alpha \in \mathbb{R}$ *such that* $\alpha \mathbf{v} \notin S$

REMARK: Always assume that the operations of vector addition & scalar multiplication are the standard definitions.

One could define these operations in other ways, but such scenarios are dealt with extensively in **Abstract Algebra**.

Showing that a Set is **not** a Vector Space (Example)

WEX 4-2-2: Show that $S := \{(x_1, x_2, 1) : x_1, x_2 \in \mathbb{R}\}$ is not a vector space.

Showing that a Set is **not** a Vector Space (Example)

WEX 4-2-2: Show that $S := \{(x_1, x_2, 1) : x_1, x_2 \in \mathbb{R}\}$ is not a vector space.

There are three ways to show this, but writing only one of them is sufficient:

(First way)

First, realize that for vector $\mathbf{x} = (x_1, x_2, x_3)$ to be in set S, $x_3 = 1$.
i.e., the 3^{rd} component of the vector must be one.

However, observe that the **zero vector** $\vec{\mathbf{0}} = (0, 0, 0) \notin S$
(Since 3^{rd} component of $\vec{\mathbf{0}}$ is zero, not one)

Therefore, $\boxed{S \text{ is } \textbf{not} \text{ a vector space}}$.

Showing that a Set is **not** a Vector Space (Example)

WEX 4-2-2: Show that $S := \{(x_1, x_2, 1) : x_1, x_2 \in \mathbb{R}\}$ is not a vector space.

There are three ways to show this, but writing only one of them is sufficient:

(Second way)

First, realize that for vector $\mathbf{x} = (x_1, x_2, x_3)$ to be in set S, $x_3 = 1$.
i.e., the 3^{rd} component of the vector must be one.

Let $\mathbf{u} = (1, 2, 1) \in S, \mathbf{v} = (-1, 0, 1) \in S$
Then, $\mathbf{u} + \mathbf{v} = (1, 2, 1) + (-1, 0, 1) = (1 + (-1), 2 + 0, 1 + 1) = (0, 2, 2) \notin S$
(Since 3^{rd} component of $(0, 2, 2)$ is two, not one)

Therefore, S is **not** a vector space.

Showing that a Set is **not** a Vector Space (Example)

WEX 4-2-2: Show that $S := \{(x_1, x_2, 1) : x_1, x_2 \in \mathbb{R}\}$ is not a vector space.

There are three ways to show this, but writing only one of them is sufficient:

(Third way)

First, realize that for vector $\mathbf{x} = (x_1, x_2, x_3)$ to be in set S, $x_3 = 1$.
i.e., the 3^{rd} component of the vector must be one.

Let $\mathbf{v} = (5, 2, 1) \in S$ and $\alpha = 4$

Then, $\alpha\mathbf{v} = 4(5, 2, 1) = (4 \cdot 5, 4 \cdot 2, 4 \cdot 1) = (20, 8, 4) \notin S$

(Since 3^{rd} component of $(20, 8, 4)$ is four, not one)

Therefore, $\boxed{S \text{ is } \textbf{not} \text{ a vector space}}$.

Showing that a Set is **not** a Vector Space (Example)

WEX 4-2-3: Show that $S := \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : a, d \in \mathbb{Q} \right\}$ is not a vector space.

Showing that a Set is **not** a Vector Space (Example)

WEX 4-2-3: Show that $S := \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : a, d \in \mathbb{Q} \right\}$ is not a vector space.

First, realize that for matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ to be in set S,

$a_{12} = a_{21} = 0$ and a_{11} & a_{22} must both be **rational**.

i.e., the (1,2)-entry & (2,1)-entry of a 2x2 matrix must both be zero,
and the (1,1)-entry & (2,2)-entry must both be **rational**.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \in S$ and $\alpha = \sqrt{5}$. Then:

$$\alpha A = \sqrt{5} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} (1)(\sqrt{5}) & (0)(\sqrt{5}) \\ (0)(\sqrt{5}) & \left(\frac{1}{3}\right)(\sqrt{5}) \end{bmatrix} = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \frac{\sqrt{5}}{3} \end{bmatrix} \notin S$$

(Since the (1,1)-entry & (2,2)-entry of αA are **not rational**)

Therefore, S is **not** a vector space.

Fin

Fin.