Subspaces

Linear Algebra

Josh Engwer

TTU

02 October 2015

Common Vector Spaces (Review)

$$
\begin{aligned}
\mathbb{R} &\equiv \text{Set of all real numbers (scalars)} \\
\mathbb{R}^2 &\equiv \text{Set of all ordered pairs (2-wide vectors)} \\
\mathbb{R}^3 &\equiv \text{Set of all ordered triples (3-wide vectors)} \\
\mathbb{R}^n &\equiv \text{Set of all ordered } n\text{-tuples } (n\text{-wide vectors}) \\
\mathbb{R}^{m\times n} &\equiv \text{Set of all } m \times n \text{ matrices} \\
\mathbb{R}^{n\times n} &\equiv \text{Set of all } n \times n \text{ square matrices} \\
P &\equiv \text{Set of all polynomials} \\
P_n &\equiv \text{Set of all polynomials of degree } n \text{ or less} \\
C[a,b] &\equiv \text{Set of all continuous functions on } [a,b] \\
C^1[a,b] &\equiv \text{Set of all differentiable functions on } [a,b] \\
C^2[a,b] &\equiv \text{Set of all twice-differentiable fcns on } [a,b] \\
C(-\infty,\infty) &\equiv \text{Set of all continuous functions on } (-\infty,\infty)
\end{aligned}
$$

REMARK: Always assume that the operations of vector addition & scalar multiplication are the standard definitions.

One could define these operations in other ways, but such scenarios are dealt with extensively in **Abstract Algebra**. (MATH 3360)

Common Vector Spaces

VECTOR SPACE	EXAMPLE "VECTORS"	"ZERO VECTOR"
$\mathbb{R}$	Scalars: $a = -3/2, b = \sqrt{2}, c = \pi$	0
$\mathbb{R}^2$	Vectors: $\mathbf{u} = (-3, 4), \mathbf{v} = (\sqrt{2}, \pi)$	$\vec{\mathbf{0}} = (0, 0)$
$\mathbb{R}^3$	Vectors: $(1, 1, 1), (\sqrt{2}, \pi, -1)$	$\vec{\mathbf{0}} = (0, 0, 0)$
$\mathbb{R}^{3\times 2}$	3×2 Matrices: $\begin{bmatrix} 1 & 2 \\ -3 & \sqrt{5} \\ -\pi & 1/6 \end{bmatrix}$	$O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\mathbb{R}^{2\times 2}$	2×2 Matrices: $\begin{bmatrix} 1 & 2 \\ -3 & \sqrt{5} \end{bmatrix}$	$O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
P_1	Polynomials: $p(t) = 3, q(t) = 4 - 2t$	$z(t) = 0 + 0t$
P_2	Polynomials: $3, 4 - 2t, 5 + t - 7t^2$	$z(t) = 0 + 0t + 0t^2$
P_3	Polynomials: $3 - 4t + 2t^2 + 5t^3$	$0 + 0t + 0t^2 + 0t^3$
$C[0, 1]$	Functions: $x^2, \sin x, \sqrt{1 + x}, \frac{1}{x-2}$	$z(x) = 0$ on $[0, 1]$
$C(-\infty, \infty)$	Functions: $x^2, \sin x, e^x, \sqrt[3]{x}, \lvert x \rvert$	$z(x) = 0$
$C^1(-\infty, \infty)$	Functions: $f(x) = x^2, \sin x, e^x$	$z(x) = 0$

REMARK: Always assume that the operations of vector addition & scalar multiplication are the standard definitions.

Subspace (Definition)

Often in applications, interesting behavior occurs not in an entire vector space, but rather a subset of it that itself acts like a vector space. Such a subset of a vector space is called a **subspace**:

Definition

(Subspace)

Let V be a vector space.
Then a nonempty set W is a **subspace** of V if the following all hold:

$W \subseteq V$ (W is a **subset** of V)
$\vec{\mathbf{0}} \in W$ (W contains the **zero vector**)

$\mathbf{u}, \mathbf{v} \in W \implies \mathbf{u} + \mathbf{v} \in W$ (W is **closed under vector addition**)
$\mathbf{v} \in W, \alpha \in \mathbb{R} \implies \alpha\mathbf{v} \in W$ (W is **closed under scalar multiplication**)

Trivial Subspaces of a Vector Space

Every vector space has two **trivial subspaces**:

Corollary

(Trivial Subspaces)

*Let V be a vector space. Then $\{\vec{\mathbf{0}}\}$ and V are the two **trivial subspaces** of V.*

REMARK: $\{\vec{\mathbf{0}}\}$ is sometimes called the **zero subspace**.

Trivial Subspaces of a Vector Space

Every vector space has two **trivial subspaces**:

Corollary

(Trivial Subspaces)

*Let V be a vector space. Then $\{\vec{\mathbf{0}}\}$ and V are the two **trivial subspaces** of V.*

<u>PROOF:</u> Since V is a vector space, it's closed under vector addition & scalar multiplication, and $\vec{\mathbf{0}} \in V$.
Moreover, a set is a subset of itself: $V \subseteq V$. Thus, V is a subspace of V.

As for the other set, let $W_0 = \{\vec{\mathbf{0}}\}$ and $\alpha \in \mathbb{R}$. Then:

$\vec{\mathbf{0}} \in W_0$ (In fact, $\vec{\mathbf{0}}$ is the <u>only</u> vector in W_0)

$W_0 \subseteq V$

$\vec{\mathbf{0}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} \in W_0 \implies W_0$ is closed under vector addition.

$\alpha\left(\vec{\mathbf{0}}\right) = \vec{\mathbf{0}} \in W_0 \implies W_0$ is closed under scalar multiplication.

Thus, $W_0 = \{\vec{\mathbf{0}}\}$ is a subspace of V. QED

The Intersection of Two Subspaces is a Subspace

Theorem

(The Intersection of Two Subspaces is a Subspace)

Let W_1, W_2 *both be subspaces of vector space* V.
Then, $W_1 \cap W_2$ *is a subspace of* V.

The Intersection of Two Subspaces is a Subspace

Theorem

(The Intersection of Two Subspaces is a Subspace)

Let W_1, W_2 both be subspaces of vector space V.
Then, $W_1 \cap W_2$ is a subspace of V.

PROOF: First observe that $W_1 \cap W_2 \subseteq W_1$, $W_1 \cap W_2 \subseteq W_2$.

Let $\alpha \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in W_1 \cap W_2$. Then:

Since W_1 is a subspace, $W_1 \subseteq V$, $\vec{\mathbf{0}} \in W_1$, $\mathbf{u}, \mathbf{v} \in W_1 \implies \mathbf{u} + \mathbf{v} \in W_1$, $\alpha\mathbf{v} \in W_1$
Since W_2 is a subspace, $W_2 \subseteq V$, $\vec{\mathbf{0}} \in W_2$, $\mathbf{u}, \mathbf{v} \in W_2 \implies \mathbf{u} + \mathbf{v} \in W_2$, $\alpha\mathbf{v} \in W_2$

Thus, $W_1 \cap W_2 \subseteq V$ since $W_1 \cap W_2 \subseteq W_1 \subseteq V$. Moreover:

$$\begin{aligned}
\vec{\mathbf{0}} \in W_1, \vec{\mathbf{0}} \in W_2 \quad &\implies \quad \vec{\mathbf{0}} \in W_1 \cap W_2 \quad \text{(Contains zero vector)} \\
\mathbf{u} + \mathbf{v} \in W_1, \mathbf{u} + \mathbf{v} \in W_2 \quad &\implies \quad \mathbf{u} + \mathbf{v} \in W_1 \cap W_2 \quad \text{(Closure under VA)} \\
\alpha\mathbf{v} \in W_1, \alpha\mathbf{v} \in W_2 \quad &\implies \quad \alpha\mathbf{v} \in W_1 \cap W_2 \quad \text{(Closure under SM)}
\end{aligned}$$

Therefore, $W_1 \cap W_2$ is a subspace of V. QED

Establishing a Set as a Subspace (Example)

WEX 4-3-1: Show that $W = \{(x, y) \in \mathbb{R}^2 : 3x - y = 0\}$ is a subspace of $\mathbb{R}^2$.

Establishing a Set as a Subspace (Example)

WEX 4-3-1: Show that $W = \{(x,y) \in \mathbb{R}^2 : 3x - y = 0\}$ is a subspace of $\mathbb{R}^2$.

Observe that, clearly, $W \subseteq \mathbb{R}^2$ (since every element of W is an element of $\mathbb{R}^2$)

Observe that the zero vector $\vec{\mathbf{0}} = (0,0) \in W$ since $3(0) - (0) = 0$

Let $\mathbf{u} = (u_1, u_2) \in W$, $\mathbf{v} = (v_1, v_2) \in W$, $\alpha \in \mathbb{R}$. Then:
Since $\mathbf{u}, \mathbf{v} \in W$, $3u_1 - u_2 = 0$ and $3v_1 - v_2 = 0$

$\implies (3u_1 - u_2) + (3v_1 - v_2) = 0 + 0$ (add the two equations)

$\implies (3u_1 + 3v_1) + (-u_2 - v_2) = 0$

$\implies 3(u_1 + v_1) - (u_2 + v_2) = 0$

$\implies (u_1 + v_1, u_2 + v_2) \in W$

$\implies \mathbf{u} + \mathbf{v} \in W$

Similarly, $3v_1 - v_2 = 0$

$\implies \alpha(3v_1 - v_2) = \alpha(0)$ (multiply both sides of equation by scalar α)

$\implies 3\alpha v_1 - \alpha v_2 = 0$

$\implies 3(\alpha v_1) - (\alpha v_2) = 0$

$\implies (\alpha v_1, \alpha v_2) \in W$

$\implies \alpha\mathbf{v} \in W$

Therefore, W is a subspace of $\mathbb{R}^2$.

Establishing that a Set is <u>**not**</u> a Subspace

Corollary

(When a Set is not a Subspace)

*W is <u>**not**</u> a subspace of vector space V if at least one of the following is true:*

- *W is not a subset of V:* $W \not\subseteq V$
- *The zero vector is not in W:* $\vec{\mathbf{0}} \notin W$
- *Closure of Vector Addition fails:* $\exists \mathbf{u}, \mathbf{v} \in W$ *such that* $\mathbf{u} + \mathbf{v} \notin W$
- *Closure of Scalar Multiplication fails:* $\exists \mathbf{v} \in W, \alpha \in \mathbb{R}$ *such that* $\alpha \mathbf{v} \notin W$

<u>REMARK:</u> Always assume that the operations of vector addition & scalar multiplication are the standard definitions.

One could define these operations in other ways, but such scenarios are dealt with extensively in **Abstract Algebra**.

Showing that a Set is **not** a Subspace (Example)

WEX 4-3-2: Show that $W = \{(x, y) \in \mathbb{R}^2 : 3x - y = 4\}$ is not a subspace of $\mathbb{R}^2$.

Showing that a Set is <u>**not**</u> a Subspace (Example)

WEX 4-3-2: Show that $W = \{(x, y) \in \mathbb{R}^2 : 3x - y = 4\}$ is not a subspace of $\mathbb{R}^2$.

There are three ways to show this, but writing only one of them is sufficient:

(First way)

Observe that, clearly, $W \subseteq \mathbb{R}^2$ (since every element of W is an element of $\mathbb{R}^2$)

Observe that the **zero vector** $\vec{\mathbf{0}} = (0, 0) \notin W$ since $3(0) - (0) = 0 \neq 4$

$\therefore$ Zero Vector is not contained in W.

Therefore, W is not a subspace of $\mathbb{R}^2$.

Showing that a Set is not a Subspace (Example)

WEX 4-3-2: Show that $W = \{(x, y) \in \mathbb{R}^2 : 3x - y = 4\}$ is not a subspace of $\mathbb{R}^2$.

There are three ways to show this, but writing only one of them is sufficient:

(Second way)

Observe that, clearly, $W \subseteq \mathbb{R}^2$ (since every element of W is an element of $\mathbb{R}^2$)

Let $\mathbf{u} = (u_1, u_2) \in W$, $\mathbf{v} = (v_1, v_2) \in W$

Then, $3u_1 - u_2 = 4$ and $3v_1 - v_2 = 4$

$\implies (3u_1 - u_2) + (3v_1 - v_2) = 4 + 4$ (add the two equations)

$\implies (3u_1 + 3v_1) + (-u_2 - v_2) = 8$

$\implies 3(u_1 + v_1) - (u_2 + v_2) = 8$

$\implies (u_1 + v_1, u_2 + v_2) \notin W$ (since RHS of previous eqn is 8, not 4)

$\implies \mathbf{u} + \mathbf{v} \notin W$

$\therefore$ Vector Addition is not closed in W.

Therefore, W is not a subspace of $\mathbb{R}^2$.

Showing that a Set is **not** a Subspace (Example)

WEX 4-3-2: Show that $W = \{(x, y) \in \mathbb{R}^2 : 3x - y = 4\}$ is not a subspace of $\mathbb{R}^2$.

There are three ways to show this, but writing only one of them is sufficient:

(Third way)

Observe that, clearly, $W \subseteq \mathbb{R}^2$ (since every element of W is an element of $\mathbb{R}^2$)

Let $\mathbf{v} = (v_1, v_2) \in W$, $\alpha \in \mathbb{R}$
Then, $3v_1 - v_2 = 4$
$\implies \alpha(3v_1 - v_2) = \alpha(4)$ (multiply equation by scalar α)
$\implies 3\alpha v_1 - \alpha v_2 = 4\alpha$
$\implies 3(\alpha v_1) - (\alpha v_2) = 4\alpha$
$\implies (\alpha v_1, \alpha v_2) \notin W$ (since RHS of previous eqn is not 4 if $\alpha \neq 1$)
$\implies \alpha\mathbf{v} \notin W$

$\therefore$ Scalar Multiplication is not closed in W.

Therefore, W is not a subspace of $\mathbb{R}^2$.

Linearity of Differentiation & Integration (Review)

Theorem

(Linearity of 1^{st}-order Derivatives)

Let $f, g \in C^1[a,b]$ *and* $\alpha \in \mathbb{R}$. *Then:*

$$[f+g]'(x) \equiv \frac{d}{dx}\Big[f(x)+g(x)\Big] = \frac{d}{dx}\Big[f(x)\Big] + \frac{d}{dx}\Big[g(x)\Big] \equiv f'(x)+g'(x)$$

$$[\alpha f]'(x) \equiv \frac{d}{dx}\Big[\alpha f(x)\Big] = \alpha\frac{d}{dx}\Big[f(x)\Big] \equiv \alpha f'(x)$$

<u>PROOF:</u>

$$[f+g]'(x) := \lim_{\Delta x\to 0}\frac{[f(x+\Delta x)+g(x+\Delta x)]-[f(x)+g(x)]}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} + \lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

$$:= f'(x)+g'(x)$$

Linearity of Differentiation & Integration (Review)

Theorem

(Linearity of 1^{st}-order Derivatives)

Let $f, g \in C^1[a,b]$ *and* $\alpha \in \mathbb{R}$. *Then:*

$$[f+g]'(x) \equiv \frac{d}{dx}\Big[f(x)+g(x)\Big] = \frac{d}{dx}\Big[f(x)\Big] + \frac{d}{dx}\Big[g(x)\Big] \equiv f'(x)+g'(x)$$

$$[\alpha f]'(x) \equiv \frac{d}{dx}\Big[\alpha f(x)\Big] = \alpha\frac{d}{dx}\Big[f(x)\Big] \equiv \alpha f'(x)$$

<u>PROOF:</u>

$$[\alpha f]'(x) := \lim_{\Delta x \to 0} \frac{[\alpha f(x+\Delta x)] - [\alpha f(x)]}{\Delta x}$$

$$= \alpha \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$:= \alpha f'(x)$$ QED

Linearity of Differentiation & Integration (Review)

Corollary

(Linearity of higher-order Derivatives)

Let $f, g \in C^k[a,b]$ *where* $k > 1$ *is an integer and* $\alpha \in \mathbb{R}$. *Then:*

$$[f+g]^{(k)}(x) \equiv \frac{d^k}{dx^k}\Big[f(x)+g(x)\Big] = \frac{d^k}{dx^k}\Big[f(x)\Big] + \frac{d^k}{dx^k}\Big[g(x)\Big] \equiv f^{(k)}(x) + g^{(k)}(x)$$

$$[\alpha f]^{(k)}(x) \equiv \frac{d^k}{dx^k}\Big[\alpha f(x)\Big] = \alpha\frac{d^k}{dx^k}\Big[f(x)\Big] \equiv \alpha f^{(k)}(x)$$

PROOF: Apply the previous theorem k times. QED

Linearity of Differentiation & Integration (Review)

Theorem

(Linearity of Definite Integrals)

Let $f, g \in C[a,b]$ *and* $\alpha \in \mathbb{R}$. *Then:*

$$\int_a^b \Big[f(x) + g(x)\Big]\, dx \quad = \quad \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

$$\int_a^b \Big[\alpha f(x)\Big]\, dx \quad = \quad \alpha \int_a^b f(x)\, dx$$

PROOF:

$$\begin{aligned}
\int_a^b \Big[f(x) + g(x)\Big]\, dx \quad &:= \quad \lim_{N\to\infty} \sum_{k=1}^{N} \Big[f(x_k^*) + g(x_k^*)\Big]\Delta_k \\
&= \quad \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k^*)\Delta_k + \lim_{N\to\infty} \sum_{k=1}^{N} g(x_k^*)\Delta_k \\
&:= \quad \int_a^b f(x)\, dx + \int_a^b g(x)\, dx
\end{aligned}$$

Linearity of Differentiation & Integration (Review)

Theorem

(Linearity of Definite Integrals)

Let $f, g \in C[a,b]$ *and* $\alpha \in \mathbb{R}$. *Then:*

$$\int_a^b \Big[f(x) + g(x)\Big]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

$$\int_a^b \Big[\alpha f(x)\Big]\, dx = \alpha \int_a^b f(x)\, dx$$

PROOF:

$$\begin{aligned} \int_a^b \Big[\alpha f(x)\Big]\, dx &:= \lim_{N\to\infty} \sum_{k=1}^{N} \alpha f(x_k^*)\Delta_k \\ &= \alpha \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k^*)\Delta_k \\ &:= \alpha \int_a^b f(x)\, dx \quad \text{QED} \end{aligned}$$

Showing that a Set is **not** a Subspace (Example)

WEX 4-3-3: Show: $W = \{p(t) \in P_2 : p'''(2) = 5\}$ is not a subspace of P_2.

Showing that a Set is **not** a Subspace (Example)

WEX 4-3-3: Show: $W = \{p(t) \in P_2 : p'''(2) = 5\}$ is not a subspace of P_2.

There are three ways to show this, but writing only one of them is sufficient:

(First way)

Observe that, clearly, $W \subseteq P_2$ (since every element of W is an element of P_2)

Observe that the **zero vector** $z(t) = 0t^2 + 0t + 0 \notin W$ since $z'''(2) = 0 \neq 5$

$\therefore$ The Zero Vector is not contained in W.

$\therefore$ W is not a subspace of P_2.

Showing that a Set is **not** a Subspace (Example)

WEX 4-3-3: Show: $W = \{p(t) \in P_2 : p'''(2) = 5\}$ is not a subspace of P_2.

There are three ways to show this, but writing only one of them is sufficient:

(Second way)

Observe that, clearly, $W \subseteq P_2$ (since every element of W is an element of P_2)

Let $p(t) \in W$, $q(t) \in W$. Then, $p'''(2) = 5$ and $q'''(2) = 5$

$\implies [p+q]'''(2) = p'''(2) + q'''(2) = 5 + 5 = 10$

$\implies p(t) + q(t) \notin W$ (since $[p+q]'''(2) \neq 5$)

$\therefore$ Vector Addition is not closed in W.

$\therefore$ W is not a subspace of P_2.

Showing that a Set is **not** a Subspace (Example)

WEX 4-3-3: Show: $W = \{p(t) \in P_2 : p'''(2) = 5\}$ is not a subspace of P_2.

There are three ways to show this, but writing only one of them is sufficient:

(Third way)

Observe that, clearly, $W \subseteq P_2$ (since every element of W is an element of P_2)

Let $p(t) \in W$ and $\alpha \neq 1$. Then, $p'''(2) = 5$

$$\implies [\alpha p]'''(2) \equiv \frac{d^3}{dt^3}\Big[\alpha p(t)\Big]\Big|_{t=2} = \alpha \frac{d^3}{dt^3}\Big[p(t)\Big]\Big|_{t=2} \equiv \alpha[p'''(2)] = 5\alpha \neq 5$$

$\implies \alpha p(t) \notin W$ (since $[\alpha p]'''(2) \neq 5$ for $\alpha \neq 1$)

$\therefore$ Scalar Multiplication is not closed in W.

$\therefore$ W is not a subspace of P_2.

Fin.