Spanning Sets & Linear Independence

Linear Algebra

Josh Engwer

TTU

05 October 2015

PART I:

LINEAR COMBINATIONS OF VECTORS

Linear Combinations of Vectors (Definition)

For the remainder of Linear Algebra & higher math courses, the notion of a **linear combination** of vectors is crucial to the development of key ideas.

Definition

(Linear Combination of Vectors)

Let V be a vector space.
Then a vector $\vec{\mathbf{u}} \in V$ is represented as a **linear combination** of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k \in V$ if $\vec{\mathbf{u}}$ can be written in the form

$$\vec{\mathbf{u}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$$

where scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$

Examples of Linear Combinations

- Let $S = \{(1,1),(2,-1),(0,4),(1,12)\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\} \subseteq \mathbb{R}^2$.
  Then $\vec{\mathbf{v}}_4$ is a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ because
  $\vec{\mathbf{v}}_4 = 3\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 + 2\vec{\mathbf{v}}_3 = 3(1,1) - (2,-1) + 2(0,4) = (1,12)$

- Let $S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 10 \end{bmatrix} \right\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} \subseteq \mathbb{R}^3$.
  Then $\vec{\mathbf{v}}_3$ is a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ because
  $\vec{\mathbf{v}}_3 = 5\vec{\mathbf{v}}_1 + (0)\vec{\mathbf{v}}_2 = 5 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (0) \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 10 \end{bmatrix}$

- Let $S = \left\{(1,1,1,1)^T, (8,8,8,8)^T\right\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\} \subseteq \mathbb{R}^4$.
  Then $\vec{\mathbf{v}}_2$ is a linear combination of $\vec{\mathbf{v}}_1$ because
  $\vec{\mathbf{v}}_2 = 8\vec{\mathbf{v}}_1 = 8(1,1,1,1)^T = (8,8,8,8)^T$
  Then $\vec{\mathbf{v}}_1$ is a linear combination of $\vec{\mathbf{v}}_2$ because
  $\vec{\mathbf{v}}_1 = \frac{1}{8}\vec{\mathbf{v}}_2 = \frac{1}{8}(8,8,8,8)^T = (1,1,1,1)^T$

Examples of Linear Combinations

- $S = \left\{ \begin{bmatrix} 3 & 4 & 8 \\ 2 & 5 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \right\} \equiv \{A_1, A_2, A_3\} \subseteq \mathbb{R}^{2\times 3}$.

  Then A_1 is a linear combination of A_2, A_3 because

  $$A_1 = (1)A_2 + 2A_3 = (1)\begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \end{bmatrix} + (2)\begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 8 \\ 2 & 5 & 5 \end{bmatrix}$$

- $S = \{2 - t^3, t^2 - t + 5, 4t^2, 6, t - t^2 - 3t^3\} \equiv \{p_1(t), p_2(t), p_3(t), p_4(t), p_5(t)\} \subseteq P_3$

  Then $p_5(t)$ is a linear combination of $p_1(t), p_2(t), p_3(t), p_4(t)$ because

  $$\begin{aligned} p_5(t) &= 3p_1(t) + (-1)p_2(t) + (0)p_3(t) - \tfrac{1}{6}p_4(t) \\ &= 3(2 - t^3) + (-1)(t^2 - t + 5) + (0)(4t^2) - \tfrac{1}{6}(6) \\ &= 6 - 3t^3 - t^2 + t - 5 + 0 - 1 \\ &= t - t^2 - 3t^3 \end{aligned}$$

- Let $S = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -7 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} \subseteq \mathbb{R}^2$

  Then $\vec{\mathbf{v}}_3$ is a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ because

  $$\vec{\mathbf{v}}_3 = (0)\vec{\mathbf{v}}_1 + (0)\vec{\mathbf{v}}_2 = (0)\begin{bmatrix} 1 \\ -3 \end{bmatrix} + (0)\begin{bmatrix} -7 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Examples that are **not** Linear Combinations

- Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\} \subseteq \mathbb{R}^2$

  Then $\vec{\mathbf{v}}_1$ is **not** a linear combination of $\vec{\mathbf{v}}_2$ (and vice-versa). Why not??

- Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} \subseteq \mathbb{R}^2$.

  Then $\vec{\mathbf{v}}_1$ is **not** a linear combination of $\vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$. Why not??

- Let $S = \left\{ \begin{bmatrix} 3 & -3 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\} \equiv \{A_1, A_2, A_3\} \subseteq \mathbb{R}^{2\times 2}$.

  Then A_3 is **not** a linear combination of A_1, A_2. Why not??

- Let $S = \{1 - t^2, t^2, -3, t^2 + 7t + 1\} \equiv \{p_1(t), p_2(t), p_3(t), p_4(t)\} \subseteq P_2$

  Then $p_4(t)$ is **not** a linear combination of $p_1(t), p_2(t), p_3(t)$. Why not??

Finding a Linear Combination (Procedure)

Proposition

(Writing a Vector as a Linear Combination of other Vectors)

TASK: *Write* $\vec{\mathbf{u}} \in V$ *as a linear combination of* $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k \in V$*.*

(1) Let $c_1, c_2, \ldots, c_k \in \mathbb{R}$ *be unknown scalars such that*

$$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{u}}$$

(2) Compute & simplify/factor LHS expression: $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$

(3) Equate both sides of equation, component by component.

(4) Solve the resulting linear system for $c_1, c_2, \ldots, c_k$ *using Gauss-Jordan on the resulting augmented matrix* $[A \mid \vec{\mathbf{u}}]$ $(A$ *is coefficient matrix of LHS*$)$

($\star$) If there are infinitely many solutions,
let the parameters $t, s, \ldots$ *be any specific values*
(e.g. Let $t = 1$, $s = 0$, $\ldots$ *)*

($\star$) If there are no solutions (i.e. linear system is inconsistent),
it's not possible to write $\vec{\mathbf{u}}$ *as a linear combination of* $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k$

Finding a Linear Combination (Example)

<u>WEX 4-4-1:</u> Write $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$

Finding a Linear Combination (Example)

WEX 4-4-1: Write $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$

$$c_1 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Finding a Linear Combination (Example)

WEX 4-4-1: Write $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$

$$c_1 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2c_1 + 3c_2 + 3c_3 \\ 3c_1 + 2c_2 + 3c_3 \\ c_1 + 3c_2 + 2c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Finding a Linear Combination (Example)

WEX 4-4-1: Write $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$

$$c_1 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2c_1 + 3c_2 + 3c_3 \\ 3c_1 + 2c_2 + 3c_3 \\ c_1 + 3c_2 + 2c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$\implies \left\{ \begin{array}{ccccccc} 2c_1 & + & 3c_2 & + & 3c_3 & = & 1 \\ 3c_1 & + & 2c_2 & + & 3c_3 & = & 1 \\ c_1 & + & 3c_2 & + & 2c_3 & = & 4 \end{array} \right.$$

Finding a Linear Combination (Example)

WEX 4-4-1: Write $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$

$$c_1 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2c_1 + 3c_2 + 3c_3 \\ 3c_1 + 2c_2 + 3c_3 \\ c_1 + 3c_2 + 2c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$\implies \begin{cases} 2c_1 + 3c_2 + 3c_3 = 1 \\ 3c_1 + 2c_2 + 3c_3 = 1 \\ c_1 + 3c_2 + 2c_3 = 4 \end{cases}$$

$$\implies \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Finding a Linear Combination (Example)

WEX 4-4-1: Write $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$

$$\implies \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Finding a Linear Combination (Example)

WEX 4-4-1: Write $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$

$$\Longrightarrow \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & 3 & 1 \\ 3 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 \end{array}\right]$$

Finding a Linear Combination (Example)

WEX 4-4-1: Write $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$

$$\implies \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$\implies \left[\begin{array}{ccc|c} 2 & 3 & 3 & 1 \\ 3 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & -8 \end{array}\right]$$

Finding a Linear Combination (Example)

WEX 4-4-1: Write $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$

$$\implies \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$\implies \left[\begin{array}{ccc|c} 2 & 3 & 3 & 1 \\ 3 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & -8 \end{array}\right]$$

$$\implies c_1 = 5, c_2 = 5, c_3 = -8$$

Finding a Linear Combination (Example)

<u>WEX 4-4-1:</u> Write $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$

$$\Longrightarrow \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & 3 & 1 \\ 3 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & -8 \end{array}\right]$$

$$\Longrightarrow c_1 = 5, c_2 = 5, c_3 = -8$$

$$\therefore \boxed{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + (-8) \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}}$$

$$\vec{\mathbf{u}} = 5\vec{\mathbf{v}}_1 + 5\vec{\mathbf{v}}_2 + (-8)\vec{\mathbf{v}}_3$$

Finding a Linear Combination (Example)

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{cc} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array} \right\}$

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{rr} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array} \right\}$

$$c_1(3t-3) + c_2(-3t+3) + c_3(t^2-4t+1) + c_4(-4t-1) = -3t^2 - 3t - 3$$

Finding a Linear Combination (Example)

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{rr} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array} \right\}$

$$c_1(3t-3) + c_2(-3t+3) + c_3(t^2-4t+1) + c_4(-4t-1) = -3t^2 - 3t - 3$$

$$\implies 3c_1t - 3c_1 - 3c_2t + 3c_2 + c_3t^2 - 4c_3t + c_3 - 4c_4t - c_4 = -3t^2 - 3t - 3$$

Finding a Linear Combination (Example)

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{rr} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array} \right\}$

$$c_1(3t-3) + c_2(-3t+3) + c_3(t^2-4t+1) + c_4(-4t-1) = -3t^2 - 3t - 3$$

$$\implies 3c_1t - 3c_1 - 3c_2t + 3c_2 + c_3t^2 - 4c_3t + c_3 - 4c_4t - c_4 = -3t^2 - 3t - 3$$

$$\implies c_3t^2 + (3c_1 - 3c_2 - 4c_3 - 4c_4)t + (-3c_1 + 3c_2 + c_3 - c_4) = -3t^2 - 3t - 3$$

Finding a Linear Combination (Example)

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{rr} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array} \right\}$

$$c_1(3t-3) + c_2(-3t+3) + c_3(t^2-4t+1) + c_4(-4t-1) = -3t^2 - 3t - 3$$

$$\implies 3c_1t - 3c_1 - 3c_2t + 3c_2 + c_3t^2 - 4c_3t + c_3 - 4c_4t - c_4 = -3t^2 - 3t - 3$$

$$\implies c_3t^2 + (3c_1 - 3c_2 - 4c_3 - 4c_4)t + (-3c_1 + 3c_2 + c_3 - c_4) = -3t^2 - 3t - 3$$

$$\implies \left\{ \begin{array}{rcrcrcrcr} & & & & c_3 & & & = & -3 \\ 3c_1 & - & 3c_2 & - & 4c_3 & - & 4c_4 & = & -3 \\ -3c_1 & + & 3c_2 & + & c_3 & - & c_4 & = & -3 \end{array} \right\}$$

Finding a Linear Combination (Example)

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{cc} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array} \right\}$

$$c_1(3t-3) + c_2(-3t+3) + c_3(t^2-4t+1) + c_4(-4t-1) = -3t^2 - 3t - 3$$

$$\implies 3c_1t - 3c_1 - 3c_2t + 3c_2 + c_3t^2 - 4c_3t + c_3 - 4c_4t - c_4 = -3t^2 - 3t - 3$$

$$\implies c_3t^2 + (3c_1 - 3c_2 - 4c_3 - 4c_4)t + (-3c_1 + 3c_2 + c_3 - c_4) = -3t^2 - 3t - 3$$

$$\implies \left\{ \begin{array}{rcrcrcrcr} & & & & c_3 & & & = & -3 \\ 3c_1 & - & 3c_2 & - & 4c_3 & - & 4c_4 & = & -3 \\ -3c_1 & + & 3c_2 & + & c_3 & - & c_4 & = & -3 \end{array} \right\}$$

$$\implies \begin{bmatrix} 0 & 0 & 1 & 0 \\ 3 & -3 & -4 & -4 \\ -3 & 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

Finding a Linear Combination (Example)

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{rr} 3t - 3, & -3t + 3, \\ t^2 - 4t + 1, & -4t - 1 \end{array} \right\}$

$$\Longrightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 3 & -3 & -4 & -4 \\ -3 & 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

Finding a Linear Combination (Example)

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{rr} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array} \right\}$

$$\Longrightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 3 & -3 & -4 & -4 \\ -3 & 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

$$\Longrightarrow \left[\begin{array}{rrrr|r} 0 & 0 & 1 & 0 & -3 \\ 3 & -3 & -4 & -4 & -3 \\ -3 & 3 & 1 & -1 & -3 \end{array} \right]$$

Finding a Linear Combination (Example)

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{rr} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array} \right\}$

$$\Longrightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 3 & -3 & -4 & -4 \\ -3 & 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

$$\Longrightarrow \left[\begin{array}{rrrr|r} 0 & 0 & 1 & 0 & -3 \\ 3 & -3 & -4 & -4 & -3 \\ -3 & 3 & 1 & -1 & -3 \end{array}\right] \sim \left[\begin{array}{rrrr|r} \boxed{1} & -1 & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 3 \end{array}\right]$$

Finding a Linear Combination (Example)

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{rr} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array} \right\}$

$$\Longrightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 3 & -3 & -4 & -4 \\ -3 & 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

$$\Longrightarrow \left[\begin{array}{rrrr|r} 0 & 0 & 1 & 0 & -3 \\ 3 & -3 & -4 & -4 & -3 \\ -3 & 3 & 1 & -1 & -3 \end{array} \right] \sim \left[\begin{array}{rrrr|r} \boxed{1} & -1 & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 3 \end{array} \right]$$

$\Longrightarrow$ Let $c_2 = \tilde{t}$. Then, $c_4 = 3, c_3 = -3, c_1 - c_2 = -1$

WEX 4-4-2:

Write $-3t^2-3t-3$ as a linear combination of $\left\{\begin{array}{cc} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array}\right\}$

$$\Longrightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 3 & -3 & -4 & -4 \\ -3 & 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

$$\Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & -3 \\ 3 & -3 & -4 & -4 & -3 \\ -3 & 3 & 1 & -1 & -3 \end{array}\right] \sim \left[\begin{array}{cccc|c} \boxed{1} & -1 & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 3 \end{array}\right]$$

$\Longrightarrow$ Let $c_2 = \widetilde{t}$. Then, $c_4 = 3, c_3 = -3, c_1 - c_2 = -1$

$$\Longrightarrow (c_1, c_2, c_3, c_4)^T = \left(\widetilde{t}-1, \widetilde{t}, -3, 3\right)^T$$

Finding a Linear Combination (Example)

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{cc} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array} \right\}$

$$\Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & -3 \\ 3 & -3 & -4 & -4 & -3 \\ -3 & 3 & 1 & -1 & -3 \end{array}\right] \sim \left[\begin{array}{cccc|c} \boxed{1} & -1 & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 3 \end{array}\right]$$

$\Longrightarrow$ Let $c_2 = \widetilde{t}$. Then, $c_4 = 3, c_3 = -3, c_1 - c_2 = -1$

$\Longrightarrow (c_1, c_2, c_3, c_4)^T = \left(\widetilde{t} - 1, \widetilde{t}, -3, 3\right)^T$

Since there's a parameter $\left(\, \widetilde{t}\, \right)$, the linear system has infinitely many solutions. But for the purposes of this problem, only one solution is needed:

Finding a Linear Combination (Example)

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{cc} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array} \right\}$

$$\Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & -3 \\ 3 & -3 & -4 & -4 & -3 \\ -3 & 3 & 1 & -1 & -3 \end{array} \right] \sim \left[\begin{array}{cccc|c} \boxed{1} & -1 & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 3 \end{array} \right]$$

$\Longrightarrow$ Let $c_2 = \widetilde{t}$. Then, $c_4 = 3, c_3 = -3, c_1 - c_2 = -1$

$$\Longrightarrow (c_1, c_2, c_3, c_4)^T = \left(\widetilde{t} - 1, \widetilde{t}, -3, 3\right)^T$$

Since there's a parameter $\left(\widetilde{t}\right)$, the linear system has infinitely many solutions. But for the purposes of this problem, only one solution is needed:

Let $\widetilde{t} = 1$. Then $(c_1, c_2, c_3, c_4)^T = (0, 1, -3, 3)^T$

Therefore,

$$\boxed{-3t^2 - 3t - 3 = (0)(3t-3) + (1)(-3t+3) + (-3)(t^2-4t+1) + (3)(-4t-1)}$$

$$r(t) = (0)p_1(t) + 1p_2(t) + (-3)p_3(t) + 3p_4(t)$$

Finding a Linear Combination (Example)

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{cc} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array} \right\}$

$$\implies \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & -3 \\ 3 & -3 & -4 & -4 & -3 \\ -3 & 3 & 1 & -1 & -3 \end{array}\right] \sim \left[\begin{array}{cccc|c} \boxed{1} & -1 & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 3 \end{array}\right]$$

$\implies$ Let $c_2 = \widetilde{t}$. Then, $c_4 = 3, c_3 = -3, c_1 - c_2 = -1$

$$\implies (c_1, c_2, c_3, c_4)^T = \left(\widetilde{t} - 1, \widetilde{t}, -3, 3\right)^T$$

Since there's a parameter $\left(\,\widetilde{t}\,\right)$, the linear system has infinitely many solutions. But for the purposes of this problem, only one solution is needed:

Let $\widetilde{t} = 0$. Then $(c_1, c_2, c_3, c_4)^T = (-1, 0, -3, 3)^T$

Therefore,

$$\boxed{-3t^2 - 3t - 3 = (-1)(3t-3) + (0)(-3t+3) + (-3)(t^2-4t+1) + (3)(-4t-1)}$$

$r(t) = (-1)p_1(t) + (0)p_2(t) + (-3)p_3(t) + 3p_4(t)$

Finding a Linear Combination (Example)

WEX 4-4-2:

Write $-3t^2 - 3t - 3$ as a linear combination of $\left\{ \begin{array}{cc} 3t-3, & -3t+3, \\ t^2-4t+1, & -4t-1 \end{array} \right\}$

$$\implies \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & -3 \\ 3 & -3 & -4 & -4 & -3 \\ -3 & 3 & 1 & -1 & -3 \end{array} \right] \sim \left[\begin{array}{cccc|c} \boxed{1} & -1 & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 3 \end{array} \right]$$

$\implies$ Let $c_2 = \widetilde{t}$. Then, $c_4 = 3, c_3 = -3, c_1 - c_2 = -1$

$\implies (c_1, c_2, c_3, c_4)^T = \left(\widetilde{t} - 1, \widetilde{t}, -3, 3\right)^T$

Since there's a parameter $\left(\widetilde{t}\right)$, the linear system has infinitely many solutions. But for the purposes of this problem, only one solution is needed:

Let $\widetilde{t} = 2$. Then $(c_1, c_2, c_3, c_4)^T = (1, 2, -3, 3)^T$

Therefore,

$$\boxed{-3t^2 - 3t - 3 = (1)(3t-3) + (2)(-3t+3) + (-3)(t^2-4t+1) + (3)(-4t-1)}$$

$r(t) = 1p_1(t) + 2p_2(t) + (-3)p_3(t) + 3p_4(t)$

Finding a Linear Combination (Example)

WEX 4-4-3:

Write $\begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

Finding a Linear Combination (Example)

WEX 4-4-3:

Write $\begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

$$c_1 \begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix} + c_2 \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$$

Finding a Linear Combination (Example)

WEX 4-4-3:

Write $\begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

$$c_1 \begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix} + c_2 \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$$

$$\implies \begin{bmatrix} -c_1 & 3c_1 \\ -3c_1 & -3c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 & -4c_2 \\ 2c_2 & 2c_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$$

WEX 4-4-3:

Write $\begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

$$c_1 \begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix} + c_2 \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$$

$$\implies \begin{bmatrix} -c_1 & 3c_1 \\ -3c_1 & -3c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 & -4c_2 \\ 2c_2 & 2c_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$$

$$\implies \begin{bmatrix} -c_1 + 3c_2 & 3c_1 - 4c_2 \\ -3c_1 + 2c_2 & -3c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$$

Finding a Linear Combination (Example)

WEX 4-4-3:

Write $\begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

$$c_1 \begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix} + c_2 \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$$

$$\implies \begin{bmatrix} -c_1 & 3c_1 \\ -3c_1 & -3c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 & -4c_2 \\ 2c_2 & 2c_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$$

$$\implies \begin{bmatrix} -c_1 + 3c_2 & 3c_1 - 4c_2 \\ -3c_1 + 2c_2 & -3c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$$

$$\implies \begin{cases} -c_1 + 3c_2 = -1 \\ 3c_1 - 4c_2 = -1 \\ -3c_1 + 2c_2 = 4 \\ -3c_1 + 2c_2 = -1 \end{cases}$$

Finding a Linear Combination (Example)

WEX 4-4-3:

Write $\begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

$$c_1 \begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix} + c_2 \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} -c_1 & 3c_1 \\ -3c_1 & -3c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 & -4c_2 \\ 2c_2 & 2c_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} -c_1 + 3c_2 & 3c_1 - 4c_2 \\ -3c_1 + 2c_2 & -3c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} -c_1 + 3c_2 = -1 \\ 3c_1 - 4c_2 = -1 \\ -3c_1 + 2c_2 = 4 \\ -3c_1 + 2c_2 = -1 \end{cases}$$

$$\Longrightarrow \begin{bmatrix} -1 & 3 \\ 3 & -4 \\ -3 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 4 \\ -1 \end{bmatrix}$$

WEX 4-4-3:

Write $\begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

$$\Longrightarrow \begin{bmatrix} -1 & 3 \\ 3 & -4 \\ -3 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 4 \\ -1 \end{bmatrix}$$

Finding a Linear Combination (Example)

WEX 4-4-3:

Write $\begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

$$\implies \begin{bmatrix} -1 & 3 \\ 3 & -4 \\ -3 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 4 \\ -1 \end{bmatrix}$$

$$\implies \left[\begin{array}{cc|c} -1 & 3 & -1 \\ 3 & -4 & -1 \\ -3 & 2 & 4 \\ -3 & 2 & -1 \end{array}\right]$$

Finding a Linear Combination (Example)

WEX 4-4-3:

Write $\begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

$$\Longrightarrow \begin{bmatrix} -1 & 3 \\ 3 & -4 \\ -3 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 4 \\ -1 \end{bmatrix}$$

$$\Longrightarrow \left[\begin{array}{cc|c} -1 & 3 & -1 \\ 3 & -4 & -1 \\ -3 & 2 & 4 \\ -3 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

WEX 4-4-3:

Write $\begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

$$\implies \begin{bmatrix} -1 & 3 \\ 3 & -4 \\ -3 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 4 \\ -1 \end{bmatrix}$$

$$\implies \left[\begin{array}{cc|c} -1 & 3 & -1 \\ 3 & -4 & -1 \\ -3 & 2 & 4 \\ -3 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

But interpreting the 3^{rd} row of the RREF yields:

$0c_1 + 0c_2 = 1 \implies 0 = 1 \leftarrow$ CONTRADICTION!

Finding a Linear Combination (Example)

WEX 4-4-3:

Write $\begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

$$\implies \begin{bmatrix} -1 & 3 \\ 3 & -4 \\ -3 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 4 \\ -1 \end{bmatrix}$$

$$\implies \left[\begin{array}{cc|c} -1 & 3 & -1 \\ 3 & -4 & -1 \\ -3 & 2 & 4 \\ -3 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

But interpreting the 3^{rd} row of the RREF yields:

$0c_1 + 0c_2 = 1 \implies 0 = 1 \leftarrow$ CONTRADICTION!

Therefore, the linear system has **no solution**.

$\therefore$ The desired linear combination is **not possible**

Finding a Linear Combination (Simplified Procedure)

Notice in the previous examples that in the resulting linear system $A\vec{\mathbf{c}} = \vec{\mathbf{u}}$, the vectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k$ always formed the columns of A.

For polynomials, form each column using the coefficients of each polynomial.

Based on this observation, the procedure can be simplified:

Proposition

(Writing a Vector as a Linear Combination of other Vectors)

TASK: Write $\vec{\mathbf{u}} \in V$ as a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k \in V$.

(1) Let $\vec{\mathbf{c}} = \{c_1, c_2, \ldots, c_k\} \subseteq \mathbb{R}$ be scalars s.t. $c_1\vec{\mathbf{v}}_1 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{u}}$

(2) Solve $A\vec{\mathbf{c}} = \vec{\mathbf{u}}$ for $\vec{\mathbf{c}}$ using Gauss-Jordan on $[A \mid \vec{\mathbf{u}}] = \left[\begin{array}{ccc|c} | & & | & | \\ \vec{\mathbf{v}}_1 & \cdots & \vec{\mathbf{v}}_k & \vec{\mathbf{u}} \\ | & & | & | \end{array}\right]$

($\star$) If there are infinitely many solutions,
let the parameters $t, s, \ldots$ be any specific values (e.g. Let $t = 1$, $s = 0$, $\ldots$)

($\star$) If there are no solutions (i.e. linear system is inconsistent),
it's not possible to write $\vec{\mathbf{u}}$ as a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k$

PART II:

SPANNING SETS OF VECTORS
LINEAR INDEPENDENCE OF VECTORS

Spanning a Vector Space (Definitions)

Definition

(Spanning Set of a Vector Space)

Let V be a vector space and $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_k\} \subseteq V$
Then the **span of** S is the **set of all linear combination** of vectors in S:

$$\text{span}(S) \equiv \text{ span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_k\} := \{c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k : c_1, c_2, \cdots, c_k \in \mathbb{R}\}$$

Moreover, S **spans** V if span$(S) = V$.
i.e. S **spans** V if every vector of V can be written as a linear combination of vectors in S.

Theorem

(A Spanning Set is a Subspace)

Let V be a vector space and $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_k\} \subseteq V$

Then span(S) is a subspace of V.
Moreover, span(S) is the smallest subspace of V that contains S.

Example Spanning Set (Procedure Motivation)

The set $S = \left\{ \begin{bmatrix} \frac{1}{4} \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{5}{2} \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix} \right\}$ spans $\mathbb{R}^2$:

Let $\vec{\mathbf{x}} = (x_1, x_2)^T \in \mathbb{R}^2$ be an **arbitrary** vector in $\mathbb{R}^2$.
Then if S spans $\mathbb{R}^2$, $\vec{\mathbf{x}}$ can be written as linear combination of vectors in S.

$$\implies c_1 \begin{bmatrix} \frac{1}{4} \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{2} \\ -\frac{5}{2} \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{where } c_1, c_2, c_3 \in \mathbb{R}$$

$$\implies \begin{bmatrix} \frac{1}{4}c_1 + \frac{1}{2}c_2 - c_3 \\ -c_1 - \frac{5}{2}c_2 + 5c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & -1 \\ -1 & -\frac{5}{2} & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\implies [A \mid \vec{\mathbf{x}}] = \left[\begin{array}{ccc|c} \frac{1}{4} & \frac{1}{2} & -1 & x_1 \\ -1 & -\frac{5}{2} & 5 & x_2 \end{array} \right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & (20x_1 + 4x_2) \\ 0 & \boxed{1} & -2 & (-2x_2 - 8x_1) \end{array} \right]$$

$$\implies (c_1, c_2, c_3)^T = (20x_1 + 4x_2, 2t - 2x_2 - 8x_1, t)^T \qquad \textbf{(parameter } t \in \mathbb{R}\textbf{)}$$

$$\implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (20x_1 + 4x_2) \begin{bmatrix} \frac{1}{4} \\ -1 \end{bmatrix} + (2t - 2x_2 - 8x_1) \begin{bmatrix} \frac{1}{2} \\ -\frac{5}{2} \end{bmatrix} + t \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

$\implies$ Every vector in V can be written as a linear combination of vectors in S.
$\therefore$ S spans $\mathbb{R}^2$

Notice that every row of RREF of matrix A has a pivot!

Example **non**-Spanning Set (Procedure Motivation)

The set $S = \left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix} \right\}$ does not span $\mathbb{R}^2$:

Let $\vec{\mathbf{x}} = (x_1, x_2)^T \in \mathbb{R}^2$ be an **arbitrary** vector in $\mathbb{R}^2$.
Then if S spans $\mathbb{R}^2$, $\vec{\mathbf{x}}$ can be written as linear combination of vectors in S.

$$\implies c_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{where } c_1, c_2 \in \mathbb{R}$$

$$\implies \begin{bmatrix} 2c_1 + 4c_2 \\ 4c_1 + 8c_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\implies [A \mid \vec{\mathbf{x}}] = \left[\begin{array}{cc|c} 2 & 4 & x_1 \\ 4 & 8 & x_2 \end{array}\right] \sim \left[\begin{array}{cc|c} \boxed{1} & 2 & \frac{1}{2}x_1 \\ 0 & 0 & (x_2 - 2x_1) \end{array}\right]$$

Now, interpreting the 2^{nd} row yields $0c_1 + 0c_2 = (x_2 - 2x_1)$

which is only true for certain vectors in $\mathbb{R}^2$ like $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ where $x_2 - 2x_1 = 0$

However, there are vectors in $\mathbb{R}^2$ like $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $x_2 - 2x_1 \neq 0$

Therefore, some vectors in $\mathbb{R}^2$ are not linear combinations of vectors in S.
Therefore, S does not span $\mathbb{R}^2$.

Notice that the 2^{nd} row of RREF of matrix A is all zeros!

Spanning Set Test (Procedure)

Proposition

(Testing whether a Set Spans a Vector Space or not)

TASK: Determine whether $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_k\}$ *spans vector space* V.

(1) Let $\vec{\mathbf{x}} \in V$ *be an* ***arbitrary*** *vector in* V *and* $c_1, c_2, \ldots, c_k \in \mathbb{R}$ *be unknown scalars such that*

$$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{x}}$$

(2) Compute & simplify/factor LHS expression: $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$

(3) Equate both sides of equation, component by component.

(4) Form ***coefficient matrix*** A *from the LHS of the resulting linear system.*

(5) Perform Gauss-Jordan Elimination on matrix A.

($\star$) If ***every row*** *of RREF(A)* ***contains a pivot****, then* S *spans* V.

($\star$) If RREF(A) ***contains a row of all zeros****, then* S *does not span* V.

Spanning Set Test (Simplified Procedure)

Fortunately, the procedure can be greatly simplified:

Proposition

(Testing whether a Set Spans a Vector Space or not)

TASK: Determine whether $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_k\}$ spans vector space V.

(1) Form matrix A with $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_k$ as its columns: $A = \begin{bmatrix} | & | & & | \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \cdots & \vec{\mathbf{v}}_k \\ | & | & & | \end{bmatrix}$

(2) Perform Gauss-Jordan Elimination on matrix A.

*($\star$) If **every row** of RREF(A) **contains a pivot**, then S spans V.*

*($\star$) If RREF(A) **contains a row of all zeros**, then S does not span V.*

Linear Independence of a Set of Vectors (Definition)

Definition

(Linear Independence & Linear Dependence of a Set of Vectors)

Let V be a vector space.
Let $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_k\} \subseteq V$
Then S is called **linearly independent** if the vector equation

$$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$$

has only the **trivial solution** (of all zeros): $c_1 = 0, c_2 = 0, \cdots, c_k = 0$

If there are also nontrivial solutions, then S is called **linearly dependent**.

Linear Dependence & Linear Combinations

Theorem

Let V be a vector space.
Let $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_k\} \subseteq V$. Then:

S is linearly dependent $\iff$ *At least one of the vectors in S can be written as a linear combination of the other vectors in S*

Linear Dependence & Linear Combinations

Theorem

Let V be a vector space.
Let $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_k\} \subseteq V$. Then:

S is linearly dependent $\iff$ *At least one of the vectors in S can be written as a linear combination of the other vectors in S*

PROOF:

$(\Leftarrow)$: Suppose at least 1 vector in S can be written as a linear combination of the other vectors in S

WLOG, assume $\vec{\mathbf{v}}_1$ can be written as a linear comb. of $\vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \cdots, \vec{\mathbf{v}}_k$.
Then, $\vec{\mathbf{v}}_1 = c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 + \cdots + c_k\vec{\mathbf{v}}_k$, where $c_2, c_3, \cdots, c_k \in \mathbb{R}$
$\implies (-1)\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$
$\implies c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$ has a nontrivial solution (since $c_1 = -1 \neq 0$)
$\implies S$ is linearly dependent

Theorem

Let V be a vector space.
Let $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_k\} \subseteq V$. Then:

S is linearly dependent $\iff$ *At least one of the vectors in S can be written as a linear combination of the other vectors in S*

PROOF:

$(\Rightarrow)$: Suppose S is linearly dependent.

Then, $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$ has a nontrivial solution.
WLOG, assume $c_1 \neq 0$. Then, $\vec{\mathbf{v}}_1 = \left(-\frac{c_2}{c_1}\right)\vec{\mathbf{v}}_2 + \left(-\frac{c_3}{c_1}\right)\vec{\mathbf{v}}_3 + \cdots + \left(-\frac{c_k}{c_1}\right)\vec{\mathbf{v}}_k$
$\implies$ $\vec{\mathbf{v}}_1$ is a linear comb. of $\vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \cdots, \vec{\mathbf{v}}_k$
$\implies$ At least one vector in S can be written as a linear combination of the other vectors in S.
QED

Examples of Linear Dependence

- Let $S = \{(1,1),(2,-1),(0,4),(1,12)\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\} \subseteq \mathbb{R}^2$.

  Then S is linearly dependent because
  $\vec{\mathbf{v}}_4 = 3\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 + 2\vec{\mathbf{v}}_3 = 3(1,1) - (2,-1) + 2(0,4) = (1,12)$

- Let $S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 10 \end{bmatrix} \right\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} \subseteq \mathbb{R}^3$.

  Then S is linearly dependent because
  $\vec{\mathbf{v}}_3 = 5\vec{\mathbf{v}}_1 + (0)\vec{\mathbf{v}}_2 = 5 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (0) \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 10 \end{bmatrix}$

- Let $S = \left\{(1,1,1,1)^T, (8,8,8,8)^T\right\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\} \subseteq \mathbb{R}^4$.

  Then S is linearly dependent because
  $\vec{\mathbf{v}}_2 = 8\vec{\mathbf{v}}_1 = 8(1,1,1,1)^T = (8,8,8,8)^T$ OR
  $\vec{\mathbf{v}}_1 = \frac{1}{8}\vec{\mathbf{v}}_2 = \frac{1}{8}(8,8,8,8)^T = (1,1,1,1)^T$

Examples of Linear Dependence

- $S = \left\{ \begin{bmatrix} 3 & 4 & 8 \\ 2 & 5 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \right\} \equiv \{A_1, A_2, A_3\} \subseteq \mathbb{R}^{2\times 3}$.

  Then S is linearly dependent because

  $$A_1 = (1)A_2 + 2A_3 = (1) \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \end{bmatrix} + (2) \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 8 \\ 2 & 5 & 5 \end{bmatrix}$$

- $S = \{2 - t^3, t^2 - t + 5, 4t^2, 6, t - t^2 - 3t^3\} \equiv \{p_1(t), p_2(t), p_3(t), p_4(t), p_5(t)\} \subseteq P_3$

  Then S is linearly dependent because

  $$\begin{aligned} p_5(t) &= 3p_1(t) + (-1)p_2(t) + (0)p_3(t) - \tfrac{1}{6}p_4(t) \\ &= 3(2 - t^3) + (-1)(t^2 - t + 5) + (0)(4t^2) - \tfrac{1}{6}(6) \\ &= 6 - 3t^3 - t^2 + t - 5 + 0 - 1 \\ &= t - t^2 - 3t^3 \end{aligned}$$

- Let $S = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -7 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} \subseteq \mathbb{R}^2$

  Then S is linearly dependent because

  $$\vec{\mathbf{v}}_3 = (0)\vec{\mathbf{v}}_1 + (0)\vec{\mathbf{v}}_2 = (0) \begin{bmatrix} 1 \\ -3 \end{bmatrix} + (0) \begin{bmatrix} -7 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linear Independence Test (Procedure)

Proposition

(Testing whether a Set of Vectors is Linearly Independent or not)

<u>TASK:</u> Determine whether $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_k\}$ is linearly independent.

(1) Let $c_1, c_2, \ldots, c_k \in \mathbb{R}$ be unknown scalars s.t. $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$
(2) Compute & simplify/factor LHS expression: $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$

(3) Equate both sides of equation, component by component.

*(4) Form **coefficient matrix** A from the LHS of the resulting linear system.*

(5) Perform Gauss-Jordan Elimination on matrix A.

*($\star$) If <u>**every column**</u> of RREF(A) <u>**contains a pivot**</u>, S is linearly independent.*

*($\star$) If RREF(A) <u>**contains column(s) without a pivot**</u>, S is linearly dependent.*
Non-pivot columns of A are linear combinations of pivot columns of A.
Such linear comb's are expressed in the non-pivot columns of RREF(A).

Linear Independence Test (Simplified Procedure)

Fortunately, the procedure can be greatly simplified:

Proposition

(Testing whether a Set of Vectors is Linearly Independent or not)

TASK: Determine whether $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_k\}$ *is linearly independent.*

(1) Form matrix A *with* $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \cdots, \vec{\mathbf{v}}_k$ *as its columns:* $A = \begin{bmatrix} | & | & & | \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \cdots & \vec{\mathbf{v}}_k \\ | & | & & | \end{bmatrix}$

(2) Perform Gauss-Jordan Elimination on matrix A.

*($\star$) If **every column** of RREF(A) **contains a pivot**,* S *is linearly independent.*

*($\star$) If RREF(A) **contains column(s) without a pivot**,* S *is linearly dependent.*
Non-pivot columns of A *are linear combinations of pivot columns of* A.
Such linear comb's are expressed in the non-pivot columns of RREF(A).

Spans & Linear Independence (Example)

WEX 4-4-4: Let $S = \left\{ \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \right\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\} \subseteq \mathbb{R}^3$

(a) Does S span $\mathbb{R}^3$? (b) Is S linearly independent or dependent?

Spans & Linear Independence (Example)

WEX 4-4-4: Let $S = \left\{ \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \right\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\} \subseteq \mathbb{R}^3$

(a) Does S span $\mathbb{R}^3$? (b) Is S linearly independent or dependent?

Let $A = \begin{bmatrix} | & | \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix}$. Then Gauss-Jordan applied to A yields:

$$\begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 \\ -4 & -3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 \\ -4 & -3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 \\ 0 & \boxed{1} \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}$$

Spans & Linear Independence (Example)

WEX 4-4-4: Let $S = \left\{ \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \right\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\} \subseteq \mathbb{R}^3$

(a) Does S span $\mathbb{R}^3$? (b) Is S linearly independent or dependent?

Let $A = \begin{bmatrix} | & | \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix}$. Then Gauss-Jordan applied to A yields:

$$\begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 \\ -4 & -3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 \\ -4 & -3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 \\ 0 & \boxed{1} \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}$$

(a) RREF(A) contains a row of zeros $\Longrightarrow$ $\boxed{S \text{ does } \underline{\text{not}} \text{ span } \mathbb{R}^3}$

(b) Every column of RREF(A) contains a pivot $\Longrightarrow$ $\boxed{S \text{ is linearly independent}}$

WEX 4-4-5: Let
$S = \{1 + 4t + t^2, 2 + 2t - t^2, -2 - 4t, 1 - 2t, -2 - 2t + t^2\} \equiv \{p_1, p_2, p_3, p_4, p_5\}$

(a) Does S span P_2? (b) Is S linearly independent or dependent?

Spans & Linear Independence (Example)

WEX 4-4-5: Let

$S = \{1 + 4t + t^2, 2 + 2t - t^2, -2 - 4t, 1 - 2t, -2 - 2t + t^2\} \equiv \{p_1, p_2, p_3, p_4, p_5\}$

(a) Does S span P_2? (b) Is S linearly independent or dependent?

Let $A = \begin{bmatrix} | & | & | & | & | \\ p_1(t) & p_2(t) & p_3(t) & p_4(t) & p_5(t) \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$.

Then:

$$\begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 0 & -2/3 & 0 & 0 \\ 0 & \boxed{1} & -2/3 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}$$

WEX 4-4-5: Let
$S = \{1+4t+t^2, 2+2t-t^2, -2-4t, 1-2t, -2-2t+t^2\} \equiv \{p_1, p_2, p_3, p_4, p_5\}$

(a) Does S span P_2? (b) Is S linearly independent or dependent?

Let $A = \begin{bmatrix} | & | & | & | & | \\ p_1(t) & p_2(t) & p_3(t) & p_4(t) & p_5(t) \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$.

Then:

$$\begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 0 & -2/3 & 0 & 0 \\ 0 & \boxed{1} & -2/3 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}$$

(a) Every row of RREF(A) contains a pivot $\implies$ $\boxed{S \text{ spans } P_2}$

(b) Columns 3 & 5 of RREF(A) contain no pivot $\implies$ $\boxed{S \text{ is linearly dependent}}$

Spans & Linear Independence (Example)

WEX 4-4-5: Let

$S = \{1 + 4t + t^2, 2 + 2t - t^2, -2 - 4t, 1 - 2t, -2 - 2t + t^2\} \equiv \{p_1, p_2, p_3, p_4, p_5\}$

(a) Does S span P_2? (b) Is S linearly independent or dependent?

Let $A = \begin{bmatrix} | & | & | & | & | \\ p_1(t) & p_2(t) & p_3(t) & p_4(t) & p_5(t) \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$.

Then:

$$\begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 0 & -2/3 & 0 & 0 \\ 0 & \boxed{1} & -2/3 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}$$

(a) Every row of RREF(A) contains a pivot $\Longrightarrow$ $\boxed{S \text{ spans } P_2}$

(b) Columns 3 & 5 of RREF(A) contain no pivot $\Longrightarrow$ $\boxed{S \text{ is linearly dependent}}$

(Column 3 of A) $= \left(-\frac{2}{3}\right)$ (1^{st} Pivot Column of A) $+ \left(-\frac{2}{3}\right)$ (2^{nd} Pivot Column of A)

$p_3(t) = \left(-\frac{2}{3}\right) p_1(t) + \left(-\frac{2}{3}\right) p_2(t)$

$-2 - 4t = \left(-\frac{2}{3}\right)(1 + 4t + t^2) + \left(-\frac{2}{3}\right)(2 + 2t - t^2)$

Spans & Linear Independence (Example)

WEX 4-4-5: Let
$S = \{1+4t+t^2, 2+2t-t^2, -2-4t, 1-2t, -2-2t+t^2\} \equiv \{p_1, p_2, p_3, p_4, p_5\}$

(a) Does S span P_2? (b) Is S linearly independent or dependent?

Let $A = \begin{bmatrix} | & | & | & | & | \\ p_1(t) & p_2(t) & p_3(t) & p_4(t) & p_5(t) \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$.

Then:

$$\begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 0 & -2/3 & 0 & 0 \\ 0 & \boxed{1} & -2/3 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}$$

(a) Every row of RREF(A) contains a pivot $\Longrightarrow$ $\boxed{S \text{ spans } P_2}$

(b) Columns 3 & 5 of RREF(A) contain no pivot $\Longrightarrow$ $\boxed{S \text{ is linearly dependent}}$

(Column 3 of A) $= \left(-\frac{2}{3}\right)$ (1^{st} Pivot Column of A)$+\left(-\frac{2}{3}\right)$ (2^{nd} Pivot Column of A)

(Column 5 of A) $= (-1)$ (2^{nd} Pivot Column of A)

$p_5(t) = (-1)\, p_2(t) \implies -2-2t+t^2 = (-1)(2+2t-t^2)$

Fin

Fin.