Vector Spaces: Basis & Dimension

Linear Algebra

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TTU

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Basis for a Vector Space

Definition

(Basis of a Vector Space)

Let V be a vector space and $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq V$
Then $\mathcal{S}$ is a **basis** for V if:

$\mathcal{S}$ spans V **AND** $\mathcal{S}$ is linearly independent

Moreover, each vector in a basis is called a **basis vector**.

Definition

(Dimension of a Vector Space)

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for vector space V.
Then the **dimension** of V is the # of basis vectors in $\mathcal{B}$: $\dim(V) = k$

Let vector space Z contain only the **zero vector**. Then $\dim(Z) := 0$

Basis (Example)

WEX 4-5-1: Let $\mathcal{S} = \left\{ \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \mathbb{R}^3$

Is $\mathcal{S}$ a basis for $\mathbb{R}^3$? If so, what is the dimension of $\mathbb{R}^3$?

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Let $A = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix}$.

Then: $A = \begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix} \xrightarrow{Gauss-Jordan} \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{bmatrix} = \text{RREF}(A)$

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RREF(A) contains a row of zeros $\implies$ $\mathcal{S}$ does <u>not</u> span $\mathbb{R}^3$

Every column of RREF(A) contains a pivot $\implies$ $\mathcal{S}$ is linearly independent

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Every column of RREF(A) contains a pivot $\implies$ $\mathcal{S}$ is linearly independent

Since $\mathcal{S}$ is does not span $\mathbb{R}^3$, $\boxed{\mathcal{S} \text{ is not a basis for } \mathbb{R}^3}$

Basis (Example)

<u>WEX 4-5-2:</u> Let $\mathcal{S} = \left\{ \begin{array}{c} 1+4t+t^2, 2+2t-t^2, \\ -2-4t, 1-2t, -2-2t+t^2 \end{array} \right\} \equiv \{p_1, p_2, p_3, p_4, p_5\}$

Is $\mathcal{S}$ a basis for P_2? If so, what is the dimension of P_2?

Basis (Example)

WEX 4-5-2: Let $\mathcal{S} = \left\{ \begin{array}{c} 1+4t+t^2, 2+2t-t^2, \\ -2-4t, 1-2t, -2-2t+t^2 \end{array} \right\} \equiv \{p_1, p_2, p_3, p_4, p_5\}$

Is $\mathcal{S}$ a basis for P_2? If so, what is the dimension of P_2?

Let $A = \begin{bmatrix} | & | & | & | & | \\ p_1(t) & p_2(t) & p_3(t) & p_4(t) & p_5(t) \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$.

Then:

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{Gauss-Jordan} \underbrace{\begin{bmatrix} \boxed{1} & 0 & -2/3 & 0 & 0 \\ 0 & \boxed{1} & -2/3 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}}_{\text{RREF}(A)}$$

Basis (Example)

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Every row of RREF(A) contains a pivot $\implies$ $\mathcal{S}$ spans P_2

Columns 3 & 5 of RREF(A) contain no pivot $\implies$ $\mathcal{S}$ is linearly dependent

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Then:

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Every row of RREF(A) contains a pivot $\Longrightarrow$ $\mathcal{S}$ spans P_2

Columns 3 & 5 of RREF(A) contain no pivot $\Longrightarrow$ $\mathcal{S}$ is linearly dependent

Since $\mathcal{S}$ is not linearly independent, $\boxed{\mathcal{S} \text{ is not a basis for } P_2}$

Basis (Example)

<u>WEX 4-5-3:</u> Let $\mathcal{S} = \left\{ \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$

Is $\mathcal{S}$ a basis for $\mathbb{R}^3$? If so, what is the dimension of $\mathbb{R}^3$?

Basis (Example)

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Let $A = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 4 & -3 & 3 \\ 0 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix}$.

Then: $A = \begin{bmatrix} 4 & -3 & 3 \\ 0 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix} \xrightarrow{Gauss-Jordan} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} = \text{RREF}(A)$

Basis (Example)

WEX 4-5-3: Let $\mathcal{S} = \left\{ \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$

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Every row of RREF(A) contains a pivot $\implies$ $\mathcal{S}$ spans $\mathbb{R}^3$

Every column of RREF(A) contains a pivot $\implies$ $\mathcal{S}$ is linearly independent

Basis (Example)

WEX 4-5-3: Let $\mathcal{S} = \left\{ \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$

Is $\mathcal{S}$ a basis for $\mathbb{R}^3$? If so, what is the dimension of $\mathbb{R}^3$?

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Then: $$A = \begin{bmatrix} 4 & -3 & 3 \\ 0 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix} \xrightarrow{Gauss-Jordan} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} = \text{RREF}(A)$$

Every row of RREF(A) contains a pivot $\implies$ $\mathcal{S}$ spans $\mathbb{R}^3$

Every column of RREF(A) contains a pivot $\implies$ $\mathcal{S}$ is linearly independent

Since $\mathcal{S}$ spans $\mathbb{R}^3$ and $\mathcal{S}$ is linearly independent, $\boxed{\mathcal{S} \text{ is a basis for } \mathbb{R}^3}$

$\dim(\mathbb{R}^3) = (\#\text{ of basis vectors in } \mathcal{S}) = \boxed{3}$

Basis (Example)

WEX 4-5-4: Let
$\mathcal{S} = \{1 + 2t - 2t^2, 1 + t - 2t^2, -3 + 3t^2\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$

Is $\mathcal{S}$ a basis for P_2? If so, what is the dimension of P_2?

WEX 4-5-4: Let

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Is $\mathcal{S}$ a basis for P_2? If so, what is the dimension of P_2?

Let $$A = \begin{bmatrix} | & | & | \\ p_1(t) & p_2(t) & p_3(t) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & 0 \\ -2 & -2 & 3 \end{bmatrix}.$$

Then: $$A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & 0 \\ -2 & -2 & 3 \end{bmatrix} \xrightarrow{Gauss-Jordan} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} = \text{RREF}(A)$$

Basis (Example)

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Let $A = \begin{bmatrix} | & | & | \\ p_1(t) & p_2(t) & p_3(t) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & 0 \\ -2 & -2 & 3 \end{bmatrix}$.

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Basis (Example)

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$\mathcal{S} = \{1 + 2t - 2t^2, 1 + t - 2t^2, -3 + 3t^2\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$

Is $\mathcal{S}$ a basis for P_2? If so, what is the dimension of P_2?

Let $$A = \begin{bmatrix} | & | & | \\ p_1(t) & p_2(t) & p_3(t) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & 0 \\ -2 & -2 & 3 \end{bmatrix}.$$

Then: $$A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & 0 \\ -2 & -2 & 3 \end{bmatrix} \xrightarrow{Gauss-Jordan} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} = \text{RREF}(A)$$

Every row of RREF(A) contains a pivot $\Longrightarrow$ $\mathcal{S}$ spans P_2

Every column of RREF(A) contains a pivot $\Longrightarrow$ $\mathcal{S}$ is linearly independent

Since $\mathcal{S}$ spans P_2 <u>and</u> $\mathcal{S}$ is linearly independent, $\boxed{\mathcal{S} \text{ is a basis for } P_2}$

$\dim(P_2) = (\#\text{ of basis vectors in } \mathcal{S}) = \boxed{3}$

Finite & Infinite Dimensional Vector Spaces

It's possible for a vector space to have **infinitely many** basis vectors:

Definition

(Finite & Infinite Dimensional Vector Spaces)

Let $\mathcal{B}$ be a basis for vector space V. Then:

V is **finite dimensional** if basis $\mathcal{B}$ contains a **finite** # of basis vectors.
V is **infinite dimensional** if basis $\mathcal{B}$ contains an **infinite** # of basis vectors.

Vector spaces $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{m\times n}, P_n$ are each finite dimensional.

Infinite dimensional vector spaces are beyond the scope of this chapter:

$C[a,b] \equiv$ Vector space of **continuous functions**
$C^1[a,b] \equiv$ Vector space of **differentiable functions**
$C^2[a,b] \equiv$ Vector space of **twice-differentiable functions**
$P \equiv$ Vector Space of **all polynomials**

Infinite dimensional vector spaces will be seen occasionally in Chapters 5 & 6.

They also show up in **Differential Equations** & **Numerical Analysis**.

Standard Basis for common Vector Spaces

Many vector spaces have a ”intuitive” basis, called the **standard basis**:

VECTOR SPACE	STANDARD BASIS	DIM.
$\mathbb{R}$	$\mathcal{E} = \{1\}$	1
$\mathbb{R}^2$	$\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$	2
$\mathbb{R}^3$	$\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$	3
$\mathbb{R}^{2\times 2}$	$\mathcal{E} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$	4
$\mathbb{R}^{2\times 3}$	$\mathcal{E} = \left\{ \begin{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix} \right\}$	6
P_1	$\mathcal{E} = \{1, t\}$	2
P_2	$\mathcal{E} = \{1, t, t^2\}$	3
P_3	$\mathcal{E} = \{1, t, t^2, t^3\}$	4

Why the Standard Basis is "Intuitive"

NOTATION: A standard basis is denoted by $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$, where $\mathbf{e}_j$ is the j^{th} standard basis vector.

The coefficients in the linear combination are simply the entries of the vector:

- $x_1 = x_1(1) = x_1\mathbf{e}_1$
- $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$
- $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$
- $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{e}_1} + a_{12} \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\mathbf{e}_2} + a_{21} \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{\mathbf{e}_3} + a_{22} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{e}_4}$
- $$\begin{aligned} a_0 + a_1t + a_2t^2 + a_3t^3 &= a_0(1) + a_1(t) + a_2(t^2) + a_3(t^3) \\ &= a_0\mathbf{e}_1 + a_1\mathbf{e}_2 + a_2\mathbf{e}_3 + a_3\mathbf{e}_4 \end{aligned}$$

Finding a Basis for a Subspace Spanned by a Set

Sometimes it's necessary to find a basis for a **subspace** spanned by a **set**:

Proposition

(Finding a Basis for a Subspace Spanned by a Set)

TASK: Find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

(1) Form matrix A with $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ as its columns: $A = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & & | \end{bmatrix}$

(2) Perform Gauss-Jordan Elimination on matrix A: $A \xrightarrow{Gauss-Jordan} \text{RREF}(A)$

(3) Basis $\mathcal{B} = \{$pivot columns of $A\}$

*NOTE: For **polynomials**, form each column of matrix A using the **coefficients** of each polynomial.*

WEX 4-5-5: Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$

(a) Find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S}$. (b) Find $\dim(\text{span}\{\mathcal{S}\})$.

Finding a Basis for a Subspace Spanned by a Set

WEX 4-5-5: Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$

(a) Find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S}$. (b) Find $\dim(\text{span}\{\mathcal{S}\})$.

Let $A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$. Then:

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{Gauss-Jordan} \underbrace{\begin{bmatrix} \boxed{1} & 0 & -2/3 & 0 & 0 \\ 0 & \boxed{1} & -2/3 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}}_{\text{RREF}(A)}$$

Finding a Basis for a Subspace Spanned by a Set

WEX 4-5-5: Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$

(a) Find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S}$. (b) Find $\dim(\text{span}\{\mathcal{S}\})$.

Let $A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$. Then:

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{Gauss-Jordan} \underbrace{\begin{bmatrix} \boxed{1} & 0 & -2/3 & 0 & 0 \\ 0 & \boxed{1} & -2/3 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}}_{\text{RREF}(A)}$$

(a) $\mathcal{B} = \{\text{pivot columns of } A\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\} = \boxed{\left\{ \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}}$

(b) $\dim(\text{span}\{\mathcal{S}\}) = (\#\text{ basis vectors in } \mathcal{B}) = \boxed{3}$

Finding a Basis for a Subspace with Parameters

Sometimes it's necessary to find a basis for a **subspace** written in terms of **parameters**:

Proposition

(Finding a Basis for a Subspace written as one vector of several parameters)

TASK: Find a basis $\mathcal{B}$ for subspace spanned by $W = \{\mathbf{w}_1 : s, t, ... \in \mathbb{R}\}$.

(1) "Undo" any vector addition by writing $\mathbf{w}_1$ as a sum of vectors, each of which has its own parameter.

(2) "Undo" any scalar multiplication by factoring out each parameter from each vector.

(3) The resulting set of vectors span W.

(4) Apply the previous procedure to this set of vectors to remove any linearly dependent vectors.

Finding a Basis for a Subspace with Parameters

WEX 4-5-6: Let $W = \left\{ \begin{bmatrix} r - s \\ s + 4t \\ r + s + t \\ 3r - t \end{bmatrix} : r, s, t \in \mathbb{R} \right\}$ be a subspace of $\mathbb{R}^4$.

(a) Find a basis $\mathcal{B}$ for W (b) Find $\dim(W)$

Finding a Basis for a Subspace with Parameters

<u>WEX 4-5-6:</u> Let $W = \left\{ \begin{bmatrix} r-s \\ s+4t \\ r+s+t \\ 3r-t \end{bmatrix} : r,s,t \in \mathbb{R} \right\}$ be a subspace of $\mathbb{R}^4$.

(a) Find a basis $\mathcal{B}$ for W (b) Find dim(W)

$$\begin{bmatrix} r-s \\ s+4t \\ r+s+t \\ 3r-t \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ r \\ 3r \end{bmatrix} + \begin{bmatrix} -s \\ s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4t \\ t \\ -t \end{bmatrix} \qquad \text{(Undo Vector Addition)}$$

$$= r\begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 4 \\ 1 \\ -1 \end{bmatrix} \qquad \text{(Undo Scalar Mult.)}$$

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$$\therefore \quad W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \\ -1 \end{bmatrix} \right\} \equiv \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

$$A = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 4 \\ 1 & 1 & 1 \\ 3 & 0 & -1 \end{bmatrix} \xrightarrow{Gauss-Jordan} \underbrace{\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}}_{\text{RREF}(A)}$$

Every column of RREF(A) has a pivot $\implies$ $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.
$\therefore$ $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.

Finding a Basis for a Subspace with Parameters

WEX 4-5-6: Let $W = \left\{ \begin{bmatrix} r - s \\ s + 4t \\ r + s + t \\ 3r - t \end{bmatrix} : r, s, t \in \mathbb{R} \right\}$ be a subspace of $\mathbb{R}^4$.

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$$\begin{bmatrix} r - s \\ s + 4t \\ r + s + t \\ 3r - t \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 4 \\ 1 \\ -1 \end{bmatrix}$$

(a) $\boxed{\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \\ -1 \end{bmatrix} \right\}}$ is a basis for W

(b) $\dim(W) = (\#\text{ basis vectors in } \mathcal{B}) = 3 \implies \boxed{\dim(W) = 3}$

Fin

Fin.