Row Space, Column Space, Null Space, Rank

Linear Algebra

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TTU

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PART I:

ROW SPACE OF A MATRIX
COLUMN SPACE OF A MATRIX
RANK OF A MATRIX

Row Space & Column Space of a Matrix (Definition)

Definition

(Row Space & Column Space of a Matrix)

Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix. Then:

- The **row space** of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A:

$$\text{RowSp}(A) := \text{ span}\{\text{rows of } A\} \subseteq \mathbb{R}^n$$

  - Why $\mathbb{R}^n$? Because each row of A has n entries.

- The **column space** of A is the subspace of $\mathbb{R}^m$ spanned by columns of A:

$$\text{ColSp}(A) := \text{ span}\{\text{columns of } A\} \subseteq \mathbb{R}^m$$

  - Why $\mathbb{R}^m$? Because each column of A has m entries.

Finding Bases for the Row Space & Column Space

Proposition

(Finding Bases for the Row Space & Column Space)

TASK: Find bases for RowSp(A) & ColSp(A) where matrix $A \in \mathbb{R}^{m \times n}$.

(1) Perform Gauss-Jordan Elimination on matrix A.

*($\star$) The **pivot rows** of **RREF**(A) form a basis for RowSp(A).*
*($\star$) The **pivot columns** of A form a basis for ColSp(A).*

WARNING #1: In general, RowSp$(A) \neq$ span{pivot rows of A}.
This may happen if **row swaps** are performed for Gauss-Jordan:

If $A = \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 \\ 0 & 0 \end{bmatrix} =$ RREF(A) Then:

RowSp$(A) =$ span$\{(1,2)\} \neq$ span$\{(0,0)\} =$ span{pivot rows of A}:

$(4,8) = 4(1,2) \in$ RowSp(A),
$(4,8) \neq k(0,0) \implies (4,8) \notin$ span{pivot rows of A}

Finding Bases for the Row Space & Column Space

Proposition

(Finding Bases for the Row Space & Column Space)

TASK: Find bases for RowSp(A) *& ColSp*(A) *where matrix* $A \in \mathbb{R}^{m \times n}$.

(1) Perform Gauss-Jordan Elimination on matrix A.

($\star$) The ***pivot rows*** *of* ***RREF***(A) *form a basis for RowSp*(A).
($\star$) The ***pivot columns*** *of* A *form a basis for ColSp*(A).

WARNING #2: In general, ColSp$(A) \neq$ ColSp[RREF(A)]. For example:

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 \\ 0 & 0 \end{bmatrix} =$ RREF(A) Then:

ColSp$(A) =$ span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and ColSp[RREF(A)] $=$ span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ But:

$\begin{bmatrix} 4 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in$ ColSp(A), $\begin{bmatrix} 4 \\ 8 \end{bmatrix} \neq k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 4 \\ 8 \end{bmatrix} \notin$ ColSp[RREF(A)]

Bases for the Row Space & Column Space (Example)

WEX 4-6-1: Let $A = \begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix}$.

Find bases for the row space & column space of A.
Find the dimension of the row space & column space of A.

Bases for the Row Space & Column Space (Example)

WEX 4-6-1: Let $A = \begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix}$.

Find bases for the row space & column space of A.
Find the dimension of the row space & column space of A.

Perform Gauss-Jordan on A:

$$\begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 \\ -4 & -3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 \\ -4 & -3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 \\ 0 & \boxed{1} \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}$$

Bases for the Row Space & Column Space (Example)

WEX 4-6-1: Let $A = \begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix}$.

Find bases for the row space & column space of A.
Find the dimension of the row space & column space of A.

Perform Gauss-Jordan on A:

$$\begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 \\ -4 & -3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 \\ -4 & -3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 \\ 0 & \boxed{1} \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}$$

$\implies$ RowSp(A) = span{pivot rows of RREF(A)} = $\boxed{\text{span}\{(1,0),(0,1)\}}$

$\implies$ dim RowSp(A) = (# basis vectors in RowSp(A)) = $\boxed{2}$

$\implies$ ColSp(A) = span{pivot columns of A} = $\boxed{\text{span}\left\{\begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}\right\}}$

$\implies$ dim ColSp(A) = (# basis vectors in ColSp(A)) = $\boxed{2}$

Bases for the Row Space & Column Space (Example)

WEX 4-6-2: Let $A = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$.

Find bases for the row space & column space of A.
Find the dimension of the row space & column space of A.

Bases for the Row Space & Column Space (Example)

<u>**WEX 4-6-2:**</u> Let $A = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$.

Find bases for the row space & column space of A.
Find the dimension of the row space & column space of A.

Perform Gauss-Jordan on A:

$$\begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 0 & -2/3 & 0 & 0 \\ 0 & \boxed{1} & -2/3 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}$$

Bases for the Row Space & Column Space (Example)

WEX 4-6-2: Let $A = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$.

Find bases for the row space & column space of A.
Find the dimension of the row space & column space of A.

Perform Gauss-Jordan on A:

$$\begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 0 & -2/3 & 0 & 0 \\ 0 & \boxed{1} & -2/3 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}$$

$\Longrightarrow$ RowSp$(A) = \boxed{\text{span}\{(1,0,-2/3,0,0),(0,1,-2/3,0,-1),(0,0,0,1,0)\}}$

$\Longrightarrow$ dim RowSp(A) = (# basis vectors in RowSp(A)) $= \boxed{3}$

$\Longrightarrow$ ColSp$(A) = \boxed{\text{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}}$

$\Longrightarrow$ dim ColSp(A) = (# basis vectors in ColSp(A)) $= \boxed{3}$

More on the Row Space & Column Space

In the previous examples, the row & column space had the same dimension! This is no accident:

Theorem

(Row Space & Column Space have the Same Dimension)

Let $A \in \mathbb{R}^{m \times n}$. *Then,* *dim ColSp*$(A) =$ *dim RowSp*(A)

PROOF: See textbook if interested. (it's long & technical)

An alternative method to find bases for the row space & column space of matrix A is to consider the **transpose** of A:

Corollary

(Finding Bases for Row Space & Column Space via Transposing)

Let $A \in \mathbb{R}^{m \times n}$. *Then:*

(i) *Row space of* A *is Column Space of* A^T*:* *RowSp*(A) = *ColSp*(A^T)

(ii) *Column space of* A *is Row Space of* A^T*:* *ColSp*(A) = *RowSp*(A^T)

This provides 2 methods to find the basis of a subspace spanned by vectors.

Finding Basis of Subspace Spanned by Many Vectors

WEX 4-6-3: Let $\mathcal{S} = \{(-1,-1,2),(1,1,3),(-2,-2,2)\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} \subseteq \mathbb{R}^3$.
Find a basis for the subspace of $\mathbb{R}^3$ spanned by $\mathcal{S}$.
Find the dimension of span$(\mathcal{S})$.

Finding Basis of Subspace Spanned by Many Vectors

WEX 4-6-3: Let $\mathcal{S} = \{(-1,-1,2),(1,1,3),(-2,-2,2)\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} \subseteq \mathbb{R}^3$.
Find a basis $\mathcal{B}$ for the subspace of $\mathbb{R}^3$ spanned by $\mathcal{S}$.
Find the dimension of span$(\mathcal{S})$.

(Method One)

Form matrix A with $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ as its **rows** & find the **row space** of A:

$$A = \begin{bmatrix} \text{---} & \vec{\mathbf{v}}_1 & \text{---} \\ \text{---} & \vec{\mathbf{v}}_2 & \text{---} \\ \text{---} & \vec{\mathbf{v}}_3 & \text{---} \end{bmatrix} = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & 3 \\ -2 & -2 & 2 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$ RowSp$(A) =$ span{pivot rows of RREF(A)} $=$ span$\{(1,1,0),(0,0,1)\}$

$\Longrightarrow$ Basis $\mathcal{B} = \{$basis vectors of RowSp$(A)\} = \boxed{\{(1,1,0),(0,0,1)\}}$

$\Longrightarrow$ dim(span$\{\mathcal{S}\}$) $=$ (# basis vectors in $\mathcal{B}$) $= \boxed{2}$

Finding Basis of Subspace Spanned by Many Vectors

WEX 4-6-3: Let $\mathcal{S} = \{(-1,-1,2),(1,1,3),(-2,-2,2)\} \equiv \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} \subseteq \mathbb{R}^3$.
Find a basis $\mathcal{B}$ for the subspace of $\mathbb{R}^3$ spanned by $\mathcal{S}$.
Find the dimension of span$(\mathcal{S})$.

(Method Two)

Form matrix A with $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ as its **columns** & find the **column space** of A:

$$A = \begin{bmatrix} | & | & | \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \vec{\mathbf{v}}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -1 & 1 & -2 \\ -1 & 1 & -2 \\ 2 & 3 & 2 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 0 & 8/5 \\ 0 & \boxed{1} & -2/5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\implies \text{ColSp}(A) = \text{span}\{\text{pivot columns of } A\} = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\implies \text{Basis } \mathcal{B} = \{\text{basis vectors of ColSp}(A)\} = \boxed{\left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}}$$

$$\implies \dim(\text{span}\{\mathcal{S}\}) = (\#\text{ basis vectors in } \mathcal{B}) = \boxed{2}$$

Both Methods give completely different Bases!!

In the previous example, $\mathcal{S} = \{(-1,-1,2),(1,1,3),(-2,-2,2)\}$

Method One: Basis $\mathcal{B}_1 = \{(1,1,0),(0,0,1)\}$

Method Two: Basis $\mathcal{B}_2 = \{(-1,-1,2),(1,1,3)\}$

At first glance, it seems unlikely that bases $\mathcal{B}_1, \mathcal{B}_2$ span the same subspace!!

But in fact, they do span the same subspace as each vector in $\mathcal{S}$ is a linear combination of vectors in each basis:

$$\mathcal{B}_1: \quad \begin{array}{rcrcr} (-1,-1,2) & = & (-1)(1,1,0) & + & (2)(0,0,1) \\ (1,1,3) & = & (1)(1,1,0) & + & (3)(0,0,1) \\ (-2,-2,2) & = & (-2)(1,1,0) & + & (2)(0,0,1) \end{array}$$

$$\mathcal{B}_2: \quad \begin{array}{rcrcr} (-1,-1,2) & = & (1)(-1,-1,2) & + & (0)(1,1,3) \\ (1,1,3) & = & (0)(-1,-1,2) & + & (1)(1,1,3) \\ (-2,-2,2) & = & \left(\frac{8}{5}\right)(-1,-1,2) & + & \left(-\frac{2}{5}\right)(1,1,3) \end{array}$$

Rank of a Matrix (Definition)

Definition

(Rank of a Matrix)

Let $A \in \mathbb{R}^{m \times n}$.
Then the **rank** of A is the dimension of the column (row) space of A:

$$\text{rank}(A) := \dim \text{ColSp}(A) = \dim \text{RowSp}(A)$$

Corollary

(Rank of a Matrix)

Let $A \in \mathbb{R}^{m \times n}$.
*Then the **rank** of A is simply the # of **pivots** in **RREF**(A).*

*Matrix A has **full row rank** $\iff$ rank$(A) = m$*
*Matrix A has **full column rank** $\iff$ rank$(A) = n$*

The 1^{st} Example Revisited in terms of Rank

<u>**WEX 4-6-4:**</u> Let $A = \begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix}$.

Find the rank of A & dimension of the row space & column space.

Perform Gauss-Jordan on A:

$$\begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 \\ -4 & -3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 \\ -4 & -3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 \\ 0 & \boxed{1} \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}$$

$\Longrightarrow$ rank$(A) =$ (# pivots in RREF(A)) $= \boxed{2}$

$\Longrightarrow$ dim RowSp$(A) =$ rank$(A) = \boxed{2}$

$\Longrightarrow$ dim ColSp$(A) =$ rank$(A) = \boxed{2}$

Since **every column** of RREF(A) has a **pivot**, $\boxed{A \text{ has } \textbf{full column rank}}$

The 2^{nd} Example Revisited in terms of Rank

WEX 4-6-5: Let $A = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$.

Find the rank of A & dimension of the row space & column space.

Perform Gauss-Jordan on A:

$$\begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 0 & -2/3 & 0 & 0 \\ 0 & \boxed{1} & -2/3 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}$$

$\implies$ rank(A) = (# pivots in RREF(A)) = $\boxed{3}$

$\implies$ dim RowSp(A) = rank(A) = $\boxed{3}$

$\implies$ dim ColSp(A) = rank(A) = $\boxed{3}$

Since **every row** of RREF(A) has a **pivot**, $\boxed{A \text{ has } \textbf{full row rank}}$

Matrix-Vector Multiplication & Linear Combinations

A matrix-vector product can be a viewed as a **linear combination** of the **columns of the matrix**:

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 a_{11} + x_2 a_{12} + x_3 a_{13} \\ x_1 a_{21} + x_2 a_{22} + x_3 a_{23} \\ x_1 a_{31} + x_2 a_{32} + x_3 a_{33} \end{bmatrix}$$

$$\left(\begin{array}{c} \text{Undo Vector Addition} \\ \text{and Scalar Multiplication} \end{array} \right) = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_1 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

—— OR WHEN MATRIX IS PARTITIONED INTO COLUMN VECTORS ——

$$A\mathbf{x} = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3$$

Column Space & Linear System Consistency

Column spaces provide qualitive information about linear systems:

Theorem

(Column Space & Linear System Consistency)

Let $A \in \mathbb{R}^{m\times n}$ *and* $\mathbf{b} \in \mathbb{R}^{n}$.
Then $m \times n$ *linear system* $A\mathbf{x} = \mathbf{b}$ *is consistent* $\iff$ $\mathbf{b} \in \mathit{ColSp}(A)$

i.e. $A\mathbf{x} = \mathbf{b}$ has solution(s) $\iff$ $\mathbf{b}$ is a linear combination of the columns of A.

For instance, if $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, then $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \text{ColSp}(A)$ since

$$[A \mid \mathbf{b}] = \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 1 & 3 \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & 0 & 5/2 \\ 0 & \boxed{1} & 1/2 \end{array}\right] = [\text{RREF}(A) \mid \mathbf{b}^*]$$

$$\implies \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \left(\frac{5}{2}\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(\frac{1}{2}\right) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\implies$ $\mathbf{b}$ is a linear combination of the columns of A

Column Space & Linear System Consistency

Column spaces provide qualitive information about linear systems:

Theorem

(Column Space & Linear System Consistency)

Let $A \in \mathbb{R}^{m \times n}$ *and* $\mathbf{b} \in \mathbb{R}^n$.
Then $m \times n$ *linear system* $A\mathbf{x} = \mathbf{b}$ *is consistent* $\iff$ $\mathbf{b} \in$ *ColSp*(A)

i.e. $A\mathbf{x} = \mathbf{b}$ has solution(s) $\iff$ $\mathbf{b}$ is a linear combination of the columns of A.

For instance, if $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, then $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \notin$ ColSp(A) since

$$[A \mid \mathbf{b}] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 3 \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 0 & 0 & -1 \end{array}\right] \leftarrow \text{CONTRADICTION!}$$

$\implies$ $\mathbf{b}$ is <u>not</u> a linear combination of the columns of A

PART II:

NULLSPACE OF A MATRIX

The Null Space of a Matrix (Definition)

Definition

(Null Space of a Matrix)

Let $A \in \mathbb{R}^{m \times n}$. Then the **null space** of A is the set of all solutions to $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$:

$$\text{NulSp}(A) := \{\vec{\mathbf{x}} \in \mathbb{R}^n : A\vec{\mathbf{x}} = \vec{\mathbf{0}}\}$$

Theorem

(Null Space is a Subspace)

Let $A \in \mathbb{R}^{m \times n}$. Then NulSp$(A)$ is a subspace of $\mathbb{R}^n$.
*The **nullity** of A is the dimension of its null space: nullity(A) := dim NulSp(A)*

The Nullspace of a Matrix (Definition)

Definition

(Nullspace of a Matrix)

Let $A \in \mathbb{R}^{m \times n}$. Then the **nullspace** of A is the set of all solutions to $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$:

$$\text{NulSp}(A) := \{\vec{\mathbf{x}} \in \mathbb{R}^n : A\vec{\mathbf{x}} = \vec{\mathbf{0}}\}$$

Theorem

(Nullspace of a Matrix is a Subspace)

Let $A \in \mathbb{R}^{m \times n}$. Then NulSp$(A)$ is a subspace of $\mathbb{R}^n$.
*The **nullity** of A is the dimension of its nullspace: nullity(A) := dim NulSp(A)*

PROOF: Clearly, $\text{NulSp}(A) \subseteq \mathbb{R}^n$ and $A\vec{\mathbf{0}} = \vec{\mathbf{0}} \implies \vec{\mathbf{0}} \in \text{NulSp}(A)$

Let $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \text{NulSp}(A)$ and $c \in \mathbb{R}$. Then $A\vec{\mathbf{u}} = \vec{\mathbf{0}}$ and $A\vec{\mathbf{v}} = \vec{\mathbf{0}}$ and
$A(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = A\vec{\mathbf{u}} + A\vec{\mathbf{v}} = \vec{\mathbf{0}} + \vec{\mathbf{0}} = \vec{\mathbf{0}}$ and $A(c\vec{\mathbf{v}}) = c(A\vec{\mathbf{v}}) = c\vec{\mathbf{0}} = \vec{\mathbf{0}}$
$\implies A(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = \vec{\mathbf{0}} \implies \vec{\mathbf{u}} + \vec{\mathbf{v}} \in \text{NulSp}(A) \implies \text{NulSp}(A)$ is closed under VA
$\implies A(c\vec{\mathbf{v}}) = \vec{\mathbf{0}} \implies c\vec{\mathbf{v}} \in \text{NulSp}(A) \implies \text{NulSp}(A)$ is closed under SM

$\therefore$ NulSp(A) is a subspace of $\mathbb{R}^n$. QED

Finding the Nullspace & Nullity of a Matrix (Procedure)

Proposition

(Finding the Nullspace & Nullity of a Matrix)

TASK: Find the nullspace & nullity of matrix $A \in \mathbb{R}^{m \times n}$.

(1) Perform Gauss-Jordan Elimination on augemented matrix $[A|\vec{\mathbf{0}}]$

(2) Assign unique parameters to the free variables.

(3) Form resulting solution $\vec{\mathbf{x}}$ to $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ by interpreting rows of $[RREF(A)|\vec{\mathbf{0}}]$

(4) "Undo" vector addition by placing each parameter into its own vector.

(5) "Undo" scalar multiplication by factoring the parameter from each vector.

($\star$) The resulting vectors form a basis for the nullspace of A.
($\star$) nullity(A) = # basis vectors for the nullspace of A.
($\star$) If the only solution to $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is $\vec{\mathbf{0}}$, then NulSp$(A) = \{\vec{\mathbf{0}}\}$ & nullity$(A) = 0$.

The 1^{st} Example Revisited in terms of Nullspace

<u>**WEX 4-6-6:**</u> Let $A = \begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix}$. Find the nullspace of A and its dimension.

The 1^{st} Example Revisited in terms of Nullspace

<u>**WEX 4-6-6:**</u> Let $A = \begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix}$. Find the nullspace of A and its dimension.

Perform Gauss-Jordan on augmented matrix $[A|\vec{\mathbf{0}}]$:

$$[A|\vec{\mathbf{0}}] = \left[\begin{array}{cc|c} 0 & 2 & 0 \\ -4 & -3 & 0 \\ 2 & 2 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{array}\right] = [\text{RREF}(A)|\vec{\mathbf{0}}]$$

The 1^{st} Example Revisited in terms of Nullspace

WEX 4-6-6: Let $A = \begin{bmatrix} 0 & 2 \\ -4 & -3 \\ 2 & 2 \end{bmatrix}$. Find the nullspace of A and its dimension.

Perform Gauss-Jordan on augmented matrix $[A|\vec{\mathbf{0}}]$:

$$[A|\vec{\mathbf{0}}] = \left[\begin{array}{cc|c} 0 & 2 & 0 \\ -4 & -3 & 0 \\ 2 & 2 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{array}\right] = [\text{RREF}(A)|\vec{\mathbf{0}}]$$

Every column of RREF(A) has a pivot $\implies$ there are <u>no</u> free variables.

Interpreting the rows of $[\text{RREF}(A)|\vec{\mathbf{0}}]$ yields: $\begin{cases} 1x_1 + 0x_2 = 0 \\ 0x_1 + 1x_2 = 0 \\ 0x_1 + 0x_2 = 0 \end{cases}$

$$\implies x_1 = 0, x_2 = 0 \implies \vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \boxed{\text{NulSp}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}}$$

Since the nullspace only contains the **zero vector**,
nullity$(A) =$ dim NulSp$(A) = \boxed{0}$

The 2^{nd} Example Revisited in terms of Nullspace

<u>**WEX 4-6-7:**</u> Let $A = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$. Find its nullspace & nullity.

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Perform Gauss-Jordan on augmented matrix $[A|\vec{\mathbf{0}}]$:

$$\left[\begin{array}{ccccc|c} 1 & 2 & -2 & 1 & -2 & 0 \\ 4 & 2 & -4 & -2 & -2 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -2/3 & 0 & 0 & 0 \\ 0 & \boxed{1} & -2/3 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \end{array}\right]$$

The 2^{nd} Example Revisited in terms of Nullspace

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Columns 1,2,4 of RREF(A) have pivots $\implies$ x_3, x_5 are free variables.

Assign unique parameters to the free variables: $x_3 = s, x_5 = t$

Interpret rows of $[\text{RREF}(A)|\vec{\mathbf{0}}]$: $\begin{cases} x_1 - \frac{2}{3}x_3 = 0 & \implies & x_1 = \frac{2}{3}s \\ x_2 - \frac{2}{3}x_3 - x_5 = 0 & \implies & x_2 = \frac{2}{3}s + t \\ x_4 = 0 & \implies & x_4 = 0 \end{cases}$

The 2^{nd} Example Revisited in terms of Nullspace

WEX 4-6-7: Let $A = \begin{bmatrix} 1 & 2 & -2 & 1 & -2 \\ 4 & 2 & -4 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$. Find its nullspace & nullity.

Perform Gauss-Jordan on augmented matrix $[A|\vec{\mathbf{0}}]$:

$$\left[\begin{array}{ccccc|c} 1 & 2 & -2 & 1 & -2 & 0 \\ 4 & 2 & -4 & -2 & -2 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -2/3 & 0 & 0 & 0 \\ 0 & \boxed{1} & -2/3 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \end{array}\right]$$

$$\Longrightarrow \vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}s \\ \frac{2}{3}s + t \\ s \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3}s \\ \frac{2}{3}s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 2/3 \\ 2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \boxed{\text{NulSp}(A) = \text{span}\left\{\begin{bmatrix} 2/3 \\ 2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}} \Longrightarrow \boxed{\text{nullity}(A) = 2}$$

Nullspace & Solving Linear Systems

Theorem

(Form of Solutions to a Non-homogeneous Linear System)

Let $A \in \mathbb{R}^{m \times n}$ *where* $m \leq n$ *(i.e.* A *is square or "short & wide" rectangular)*

Let ***non-homogeneous*** *linear system* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *have* ***infinitely many*** *solutions.*

Then, the solution to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *is* $\vec{\mathbf{x}} = \vec{\mathbf{x}}_p + \vec{\mathbf{x}}_h$ *where*

$\vec{\mathbf{x}}_p$ *is the* ***particular solution*** *to* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$

$\vec{\mathbf{x}}_h$ *is the* ***homogeneous solution*** *to* $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$

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Let $A \in \mathbb{R}^{m \times n}$ where $m \leq n$ (i.e. A is square or "short & wide" rectangular)

*Let **non-homogeneous** linear system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ have **infinitely many** solutions.*

Then, the solution to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is $\vec{\mathbf{x}} = \vec{\mathbf{x}}_p + \vec{\mathbf{x}}_h$ where

*$\vec{\mathbf{x}}_p$ is the **particular solution** to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$*

*$\vec{\mathbf{x}}_h$ is the **homogeneous solution** to $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$*

PROOF: Let $\vec{\mathbf{x}}$ and $\vec{\mathbf{x}}_p$ solve the **non-homogeneous** linear system.

Then, $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and $\vec{\mathbf{x}}_p = \vec{\mathbf{b}}$

Let $\vec{\mathbf{x}}_h = \vec{\mathbf{x}} - \vec{\mathbf{x}}_p$. Then, $A\vec{\mathbf{x}}_h = A(\vec{\mathbf{x}} - \vec{\mathbf{x}}_p) = A\vec{\mathbf{x}} - A\vec{\mathbf{x}}_p = \vec{\mathbf{b}} - \vec{\mathbf{b}} = \vec{\mathbf{0}}$

$\implies A\vec{\mathbf{x}}_h = \vec{\mathbf{0}}$

$\implies \vec{\mathbf{x}} = \vec{\mathbf{x}}_p + \vec{\mathbf{x}}_h$ QED

Nullspace & Solving Linear Systems (Example)

WEX 4-6-8: Find all solution vectors of non-homongeneous linear system

$$\begin{cases} x_1 + 2x_2 - 2x_3 + x_4 - 2x_5 = 6 \\ 4x_1 + 2x_2 - 4x_3 - 2x_4 - 2x_5 = 6 \\ x_1 - x_2 + x_5 = 3 \end{cases}$$

in terms of $\vec{\mathbf{x}}_p$ & $\vec{\mathbf{x}}_h$.

Nullspace & Solving Linear Systems (Example)

WEX 4-6-8: Find all solution vectors of non-homongeneous linear system

$$\begin{cases} x_1 + 2x_2 - 2x_3 + x_4 - 2x_5 = 6 \\ 4x_1 + 2x_2 - 4x_3 - 2x_4 - 2x_5 = 6 \\ x_1 - x_2 + x_5 = 3 \end{cases}$$

in terms of $\vec{\mathbf{x}}_p$ & $\vec{\mathbf{x}}_h$.

Perform Gauss-Jordan on augmented matrix $[A|\vec{\mathbf{b}}]$:

$$[A|\vec{\mathbf{b}}] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & 1 & -2 & 6 \\ 4 & 2 & -4 & -2 & -2 & 6 \\ 1 & -1 & 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -2/3 & 0 & 0 & 3 \\ 0 & \boxed{1} & -2/3 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 3 \end{array}\right]$$

Nullspace & Solving Linear Systems (Example)

WEX 4-6-8: Find all solution vectors of non-homongeneous linear system

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Columns 1,2,4 of RREF(A) have pivots $\implies x_3, x_5$ are free variables.

Assign unique parameters to the free variables: $x_3 = s, x_5 = t$

Interpret rows:
$$\begin{cases} x_1 - \frac{2}{3}x_3 = 3 & \implies & x_1 = 3 + \frac{2}{3}s \\ x_2 - \frac{2}{3}x_3 - x_5 = 0 & \implies & x_2 = \frac{2}{3}s + t \\ x_4 = 3 & \implies & x_4 = 3 \end{cases}$$

Nullspace & Solving Linear Systems (Example)

WEX 4-6-8: Find all solution vectors of non-homongeneous linear system

$$\begin{cases} x_1 + 2x_2 - 2x_3 + x_4 - 2x_5 = 6 \\ 4x_1 + 2x_2 - 4x_3 - 2x_4 - 2x_5 = 6 \\ x_1 - x_2 + x_5 = 3 \end{cases}$$

in terms of $\vec{\mathbf{x}}_p$ & $\vec{\mathbf{x}}_h$.

Perform Gauss-Jordan on augmented matrix $[A|\vec{\mathbf{b}}]$:

$$[A|\vec{\mathbf{b}}] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & 1 & -2 & 6 \\ 4 & 2 & -4 & -2 & -2 & 6 \\ 1 & -1 & 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -2/3 & 0 & 0 & 3 \\ 0 & \boxed{1} & -2/3 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 3 \end{array}\right]$$

Interpret rows:
$$\begin{cases} x_1 - \frac{2}{3}x_3 = 3 & \Longrightarrow & x_1 = 3 + \frac{2}{3}s \\ x_2 - \frac{2}{3}x_3 - x_5 = 0 & \Longrightarrow & x_2 = \frac{2}{3}s + t \\ x_4 = 3 & \Longrightarrow & x_4 = 3 \end{cases}$$

$$\vec{\mathbf{x}} = \begin{bmatrix} 3 + \frac{2}{3}s \\ \frac{2}{3}s + t \\ s \\ 3 \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{2}{3}s \\ \frac{2}{3}s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2/3 \\ 2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Nullspace & Solving Linear Systems (Example)

WEX 4-6-8: Find all solution vectors of non-homongeneous linear system

$$\begin{cases} x_1 + 2x_2 - 2x_3 + x_4 - 2x_5 = 6 \\ 4x_1 + 2x_2 - 4x_3 - 2x_4 - 2x_5 = 6 \\ x_1 - x_2 + x_5 = 3 \end{cases}$$

in terms of $\vec{\mathbf{x}}_p$ & $\vec{\mathbf{x}}_h$.

Perform Gauss-Jordan on augmented matrix $[A|\vec{\mathbf{b}}]$:

$$[A|\vec{\mathbf{b}}] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & 1 & -2 & 6 \\ 4 & 2 & -4 & -2 & -2 & 6 \\ 1 & -1 & 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -2/3 & 0 & 0 & 3 \\ 0 & \boxed{1} & -2/3 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 3 \end{array}\right]$$

$$\boxed{\vec{\mathbf{x}} = \underbrace{\begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}}_{\vec{\mathbf{x}}_p} + s \underbrace{\begin{bmatrix} 2/3 \\ 2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{\mathbf{x}}_h} + t \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{\mathbf{x}}_h}}$$

PART III:

EQUIVALENT CONDITIONS FOR VARIOUS TYPES OF MATRICES

Equivalent Conditions for **Invertible** Square Matrices

Theorem

(Equivalent Conditions for Invertible Square Matrices)

Let $A \in \mathbb{R}^{n\times n}$ *be a* ***square*** *matrix. Then the following are equivalent:*

- *RREF*(A) *has* n *pivots*
- *rank*$(A) = n$
- A *has full row rank & full column rank*
- *The rows of* A *are linearly independent. Ditto for the columns of* A*.*
- *dim RowSp*$(A) = n$
- *dim ColSp*$(A) = n$
- *nullity*$(A) = 0$ *&* *NulSp*$(A) = \{\vec{\mathbf{0}}\}$
- *Linear system* $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ *has only the trivial solution* $\vec{\mathbf{x}} = \vec{\mathbf{0}}$
- *Linear system* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *has unique solution* $A^{-1}\vec{\mathbf{b}}$ *for every* $\vec{\mathbf{b}} \in \mathbb{R}^n$

- A *is invertible (non-singular)*
- *det*$(A) \neq 0$

Equivalent Conditions for **Singular** Square Matrices

Theorem

(Equivalent Conditions for Singular Square Matrices)

Let $A \in \mathbb{R}^{n \times n}$ be a ***square*** *matrix and $r < n$.*
Then the following are equivalent:

- *RREF*(A) *has r pivots*
- *rank*$(A) = r$
- *The rows of A are linearly dependent. Ditto for the columns of A.*
- *dim RowSp*$(A) = r$
- *dim ColSp*$(A) = r$
- *nullity*$(A) = n - r$
- *Linear system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has infinitely solutions $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h$*
- *Linear system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has infinitely many solutions only if $\vec{\mathbf{b}} \in$ ColSp(A)*
- *Linear system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has no solution only if $\vec{\mathbf{b}} \notin$ ColSp(A)*

- *A is not invertible (singular)*
- *det*$(A) = 0$

Equivalent Conditions for "**Short & Wide**" Matrices

Theorem

(Equivalent Conditions for "Short & Wide" Rectangular Matrices)

Let $A \in \mathbb{R}^{m \times n}$ be a "short & wide" rectangular matrix $(m < n)$.
Then the following are equivalent:

- *RREF(A) has m pivots*
- *rank$(A) = m$*
- *A has full row rank*
- *The rows of A are linearly independent.*
- *dim RowSp$(A) = m$*
- *dim ColSP$(A) = m$*
- *nullity$(A) = n - m$*
- *Linear system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has infinitely solutions $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h$*
- *Linear system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has infinitely solutions $\vec{\mathbf{x}} = \vec{\mathbf{x}}_p + \vec{\mathbf{x}}_h$*

Equivalent Conditions for "**Tall & Thin**" Matrices

Theorem

(Equivalent Conditions for "Tall & Thin" Rectangular Matrices)

Let $A \in \mathbb{R}^{m \times n}$ be a "tall & thin" rectangular matrix $(m > n)$.
Then the following are equivalent:

- *RREF*(A) *has* n *pivots*
- *rank*$(A) = n$
- A *has full column rank*
- *The columns of* A *are linearly independent.*
- *dim RowSp*$(A) = n$
- *dim ColSp*$(A) = n$
- *nullity*$(A) = 0$ & *NulSp*$(A) = \{\vec{\mathbf{0}}\}$
- *Linear system* $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ *has only the trivial solution* $\vec{\mathbf{x}} = \vec{\mathbf{0}}$
- *Linear system* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *has unique solution only if* $\vec{\mathbf{b}} \in$ *ColSp*(A)
- *Linear system* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *has no solution only if* $\vec{\mathbf{b}} \notin$ *ColSp*(A)

Equivalent Conditions for **Rectangular** Matrices

Theorem

(Equivalent Conditions for All Rectangular Matrices)

Let $A \in \mathbb{R}^{m \times n}$ *be a rectangular matrix and* $r < \min\{m, n\}$ *(i.e.* $r < m$ *&* $r < n$*)*
Then the following are equivalent:

- *RREF*(A) *has* r *pivots*
- *rank*$(A) = r$
- *dim ColSp*$(A) = r$
- *dim RowSp*$(A) = r$
- *nullity*$(A) = n - r$
- *Linear system* $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ *has infinitely solutions* $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h$
- *Linear system* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *has zero, one, or infinitely many solutions*

Fin

Fin.