Vectors: Norms, Dot Products, Projections

Linear Algebra

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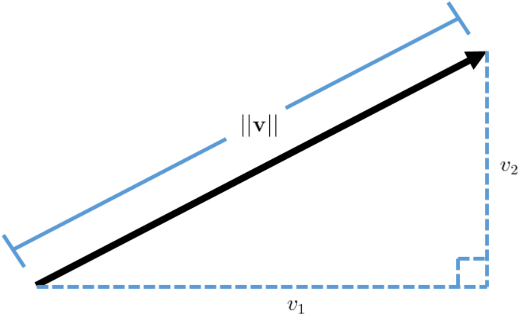
TTU

26 October 2015

Norms of Vectors in $\mathbb{R}^2$ (Definition)

The **norm** of a vector is simply its length (AKA magnitude):



Definition

The **norm** of vector $\mathbf{v} = (v_1, v_2)^T \in \mathbb{R}^2$ is defined to be

$$||\mathbf{v}|| := \sqrt{v_1^2 + v_2^2}$$

Norms of Vectors in $\mathbb{R}^3$ & $\mathbb{R}^n$ (Definition)

The **norm** can be extended to vectors in higher dimensions:

Definition

The **norm** of vector $\mathbf{v} = (v_1, v_2, v_3)^T \in \mathbb{R}^3$ is defined to be

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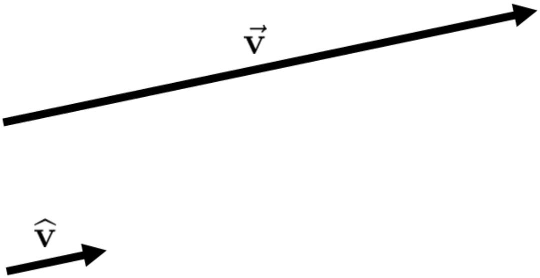
Definition

The **norm** of vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)^T \in \mathbb{R}^n$ is defined to be

$$||\mathbf{v}|| := \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Unit Vectors & Direction Vectors



Definition

A **unit vector** $\widehat{\mathbf{v}}$ is a vector with **norm one**.
A **unit vector** (AKA **direction vector**) for vector $\mathbf{v}$ is defined to be

$$\widehat{\mathbf{v}} := \frac{\mathbf{v}}{||\mathbf{v}||}$$

Dot Product in $\mathbb{R}^2$ & $\mathbb{R}^3$ (Definition)

Definition

(Dot Product in $\mathbb{R}^2$)

The **dot product** of vectors $\mathbf{v} = (v_1, v_2)^T$ and $\mathbf{w} = (w_1, w_2)^T$ is defined by:

$$\mathbf{v} \cdot \mathbf{w} := \mathbf{v}^T \mathbf{w} = \sum_{k=1}^{2} v_k w_k = v_1 w_1 + v_2 w_2$$

Definition

(Dot Product in $\mathbb{R}^3$)

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REMARK: Notice that the dot product of two vectors is a **scalar**.

Dot Product in $\mathbb{R}^n$ (Definition)

The dot product operation can be extended to vectors in higher dimensions:

Definition

(Dot Product in $\mathbb{R}^n$)

The **dot product** of vectors $\mathbf{v} = (v_1, v_2, \cdots, v_n)^T$ and $\mathbf{w} = (w_1, w_2, \cdots, w_n)^T$ is:

$$\mathbf{v} \cdot \mathbf{w} := \mathbf{v}^T \mathbf{w} = \sum_{k=1}^{n} v_k w_k = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

<u>REMARK:</u> Notice that the dot product of two vectors is a **scalar**.

Dot Product (Properties)

Corollary

(Properties of Dot Products)

Let vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and scalar* $\alpha \in \mathbb{R}$. *Then:*

(*DP1*)	$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$	*Commutativity of Dot Product*
(*DP2*)	$\alpha(\mathbf{v} \cdot \mathbf{w}) = (\alpha\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (\alpha\mathbf{w})$	*Associativity of Dot Product*
(*DP3*)	$\vec{\mathbf{0}} \cdot \mathbf{v} = \mathbf{v} \cdot \vec{\mathbf{0}} = 0$	*Dot Product with* $\vec{\mathbf{0}}$ *is Zero Scalar*
(*DP4*)	$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$	*Distributivity of Dot Product over VA*
(*DP5*)	$\mathbf{v} \cdot \mathbf{v} = \|\|\mathbf{v}\|\|^2$	*Dot Product-Norm Relationship*

Dot Product (Properties)

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(*DP5*)	$\mathbf{v} \cdot \mathbf{v} = \|\|\mathbf{v}\|\|^2$	*Dot Product-Norm Relationship*

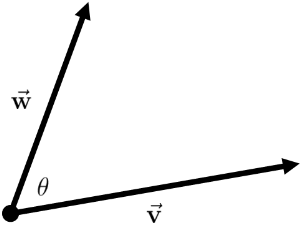
PROOF: Let $\mathbf{v} = (v_1, v_2, \cdots, v_n)^T$ and $\mathbf{w} = (w_1, w_2, \cdots, w_n)^T$. Then:

(DP1) $\mathbf{v} \cdot \mathbf{w} := v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = w_1 v_1 + w_2 v_2 + \cdots + w_n v_n := \mathbf{w} \cdot \mathbf{v}$

(DP5) $\mathbf{v} \cdot \mathbf{v} := v_1^2 + v_2^2 + \cdots + v_n^2 = \left(\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}\right)^2 := ||\mathbf{v}||^2$

QED

Dot Product (Coordinate-Free Definition)



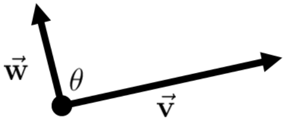
Definition

Let θ be the smallest positive angle between vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Then:

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||||\mathbf{w}|| \cos \theta \qquad \text{where } \theta \in [0, \pi]$$

- Alternative notation for the angle between vectors $\mathbf{v}, \mathbf{w}$: θ_{vw}

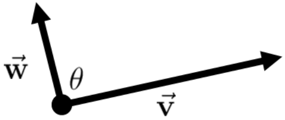
Dot Product (Orthogonality)



Theorem

Vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *are* ***orthogonal*** $\iff \mathbf{v} \perp \mathbf{w} \iff \mathbf{v} \cdot \mathbf{w} = 0$

Dot Product (Orthogonality)



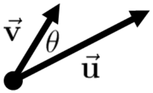
Theorem

Vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *are* ***orthogonal*** $\iff \mathbf{v} \perp \mathbf{w} \iff \mathbf{v} \cdot \mathbf{w} = 0$

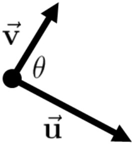
PROOF:

$\mathbf{v}, \mathbf{w}$ are **orthogonal** $\iff \theta = \pi/2 \iff \mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||||\mathbf{w}|| \cos(\pi/2) = 0$ QED

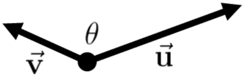
Dot Product (Geometric Interpretation)



θ is acute
$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} > 0$



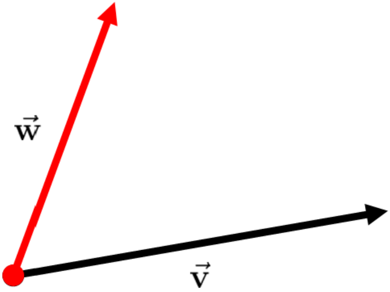
θ is 90°
$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$



θ is obtuse
$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} < 0$

Orthogonal Projection onto a Vector (Example 1)

Project $\mathbf{w}$ onto $\mathbf{v}$.



Drop perpendicular line from $\mathbf{w}$ to $\mathbf{v}$.

Orthogonal Projection onto a Vector (Example 1)

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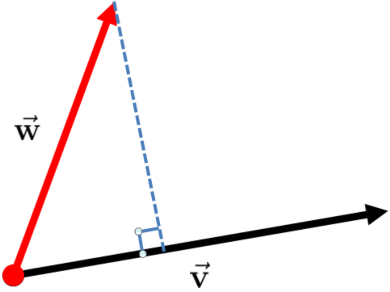

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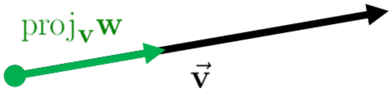
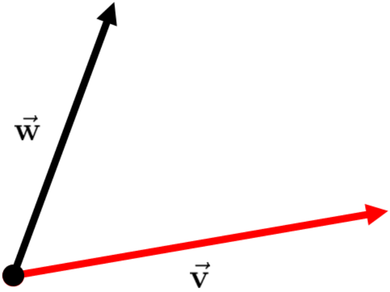


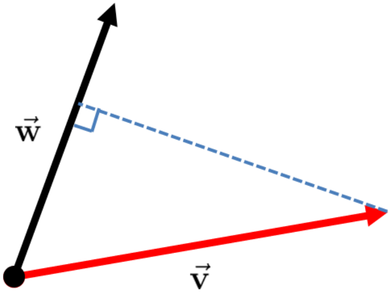
Orthogonal Projection onto a Vector (Example 2)

Project $\mathbf{v}$ onto $\mathbf{w}$.



Orthogonal Projection onto a Vector (Example 2)

Project $\mathbf{v}$ onto $\mathbf{w}$.



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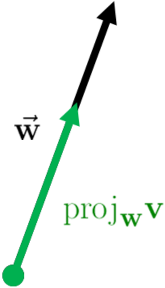
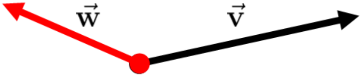
Orthogonal Projection onto a Vector (Example 2)

Project $\mathbf{v}$ onto $\mathbf{w}$.



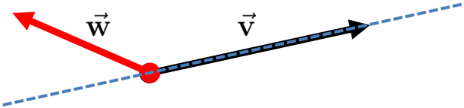
Orthogonal Projection onto a Vector (Example 3)

Project $\mathbf{w}$ onto $\mathbf{v}$.



Orthogonal Projection onto a Vector (Example 3)

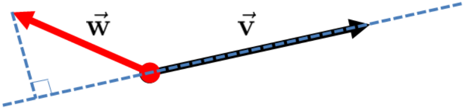
Project $\mathbf{w}$ onto $\mathbf{v}$.



Draw line extension through $\mathbf{v}$.

Orthogonal Projection onto a Vector (Example 3)

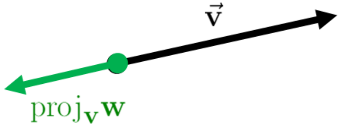
Project $\mathbf{w}$ onto $\mathbf{v}$.



Drop perpendicular line from $\mathbf{w}$ to line extension.

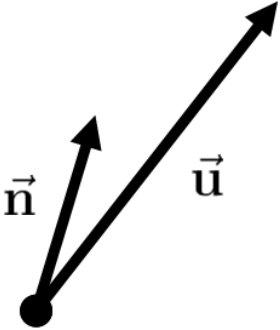
Orthogonal Projection onto a Vector (Example 3)

Project $\mathbf{w}$ onto $\mathbf{v}$.



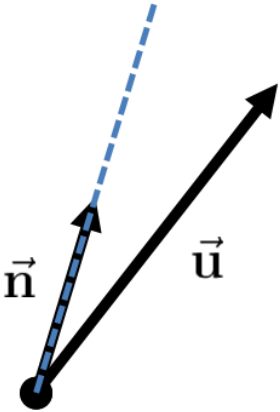
Orthogonal Projection onto a Vector (Example 4)

Project $\mathbf{u}$ onto $\mathbf{n}$.



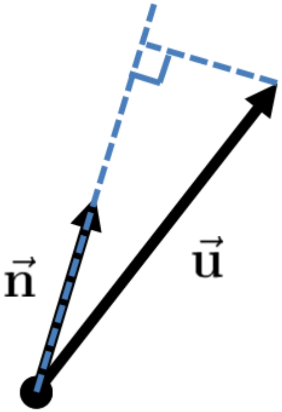
Orthogonal Projection onto a Vector (Example 4)

Project $\mathbf{u}$ onto $\mathbf{n}$.



Draw line extension through $\mathbf{n}$.

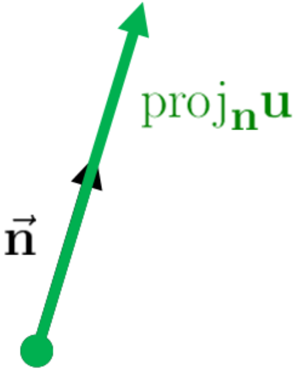
Project $\mathbf{u}$ onto $\mathbf{n}$.



Drop perpendicular line from $\mathbf{u}$ to line extension.

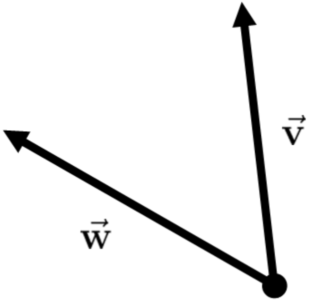
Orthogonal Projection onto a Vector (Example 4)

Project $\mathbf{u}$ onto $\mathbf{n}$.



Orthogonal Projection onto a Vector (Derivation)

Determine a formula for $\text{proj}_{\mathbf{w}}\mathbf{v}$, the **projection** of vector $\mathbf{v}$ onto vector $\mathbf{w}$.



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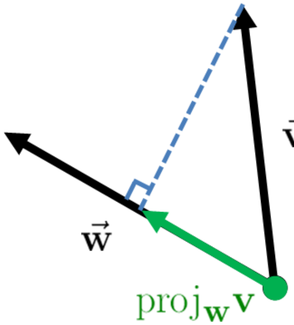
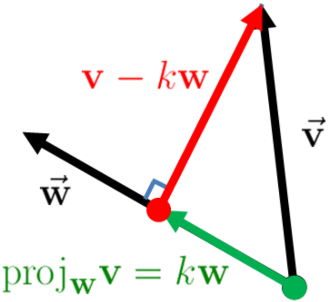


Notice that $(\text{proj}_{\mathbf{w}}\mathbf{v}) \; || \; \mathbf{w} \implies \text{proj}_{\mathbf{w}}\mathbf{v} = k\mathbf{w}$, where $k \in \mathbb{R}$.

Orthogonal Projection onto a Vector (Derivation)

Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



Form vector $\mathbf{v} - k\mathbf{w}$.

Orthogonal Projection onto a Vector (Derivation)

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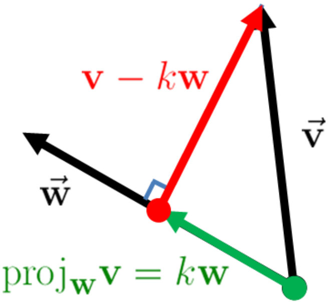
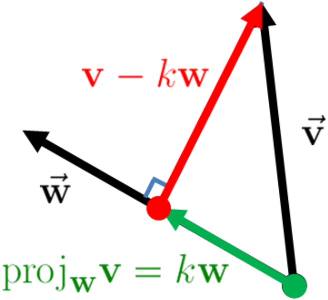


Notice that $(\mathbf{v} - k\mathbf{w}) \perp \mathbf{w}$.

Orthogonal Projection onto a Vector (Derivation)

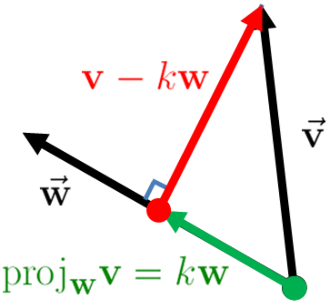
Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



$$\implies (\mathbf{v} - k\mathbf{w}) \cdot \mathbf{w} = 0$$

Orthogonal Projection onto a Vector (Derivation)

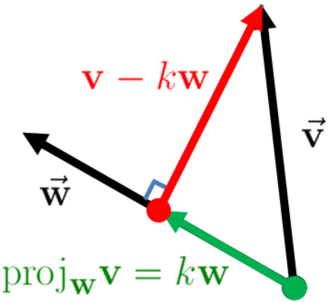
Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



$$\implies \mathbf{v} \cdot \mathbf{w} - (k\mathbf{w}) \cdot \mathbf{w} = 0$$

Orthogonal Projection onto a Vector (Derivation)

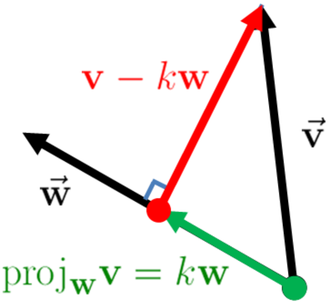
Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



$$\implies \mathbf{v} \cdot \mathbf{w} - k(\mathbf{w} \cdot \mathbf{w}) = 0$$

Orthogonal Projection onto a Vector (Derivation)

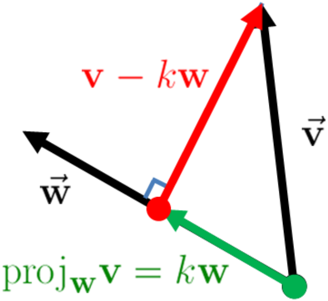
Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



$$\implies \mathbf{v} \cdot \mathbf{w} = k(\mathbf{w} \cdot \mathbf{w})$$

Orthogonal Projection onto a Vector (Derivation)

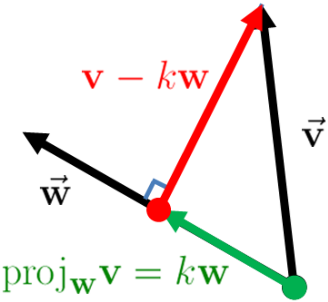
Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



$$\implies k = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}$$

Orthogonal Projection onto a Vector (Derivation)

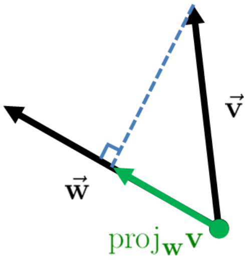
Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



$$\implies k = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}$$

$$\implies \text{proj}_{\mathbf{w}}\mathbf{v} = k\mathbf{w} = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right)\mathbf{w}$$

Orthogonal Projection onto a Vector (Formula)



Definition

(Orthogonal Projection onto a Vector)

Let vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.
Then the **(orthogonal) projection of $\mathbf{v}$ onto $\mathbf{w}$** is defined by:

$$\text{proj}_{\mathbf{w}}\mathbf{v} := \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w} = \left(\frac{\mathbf{v}^T\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\right) \mathbf{w}$$

Fin

Fin.