Inner Product Spaces

Linear Algebra

Josh Engwer

TTU

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Inner Product Space (Definition)

An **inner product** is the notion of a dot product for general vector spaces:

Definition

(Inner Product)

Let V be a vector space. Let vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and let scalar $\alpha \in \mathbb{R}$.

An **inner product** on V is a function $\langle \cdot, \cdot \rangle : V \to \mathbb{R}$ satisfying the following:

(IPS1)	$\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$	Non-negativity of Self-Inner Product
(IPS2)	$\langle \mathbf{u}, \mathbf{u} \rangle = 0 \iff \mathbf{u} = \vec{\mathbf{0}}$	Only $\vec{\mathbf{0}}$ has Self-Inner Product of Zero
(IPS3)	$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$	Commutativity of Inner Product
(IPS4)	$\langle \alpha\mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$	Inner Product of SM is SM of Inner Produc
(IPS5)	$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$	Distributivity of VA over Inner Product

(IPS1)-(IPS5) are called the **inner product axioms**.

Definition

(Inner Product Space)

A vector space V with an inner product $\langle \cdot, \cdot \rangle$ is called an **inner product space**.
A compact notation for an inner product space is: $(V, \langle \cdot, \cdot \rangle)$

Inner Product Spaces (Standard Examples)

INNER PRODUCT SPACE	PROTOTYPE "VECTORS"
$\mathbb{R}^n$	$\mathbf{u} = (u_1, \ldots, u_n)^T, \ \mathbf{v} = (v_1, \ldots, v_n)^T$
$\mathbb{R}^{m \times n}$	$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \ B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$
P_n	$p(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n$ $q(t) = q_0 + q_1 t + q_2 t^2 + \cdots + q_n t^n$ scalars $t_1, \ldots, t_{n+1} \in \mathbb{R}$
$C[a, b]$	$f(x), \ g(x)$

Inner Product Spaces (Standard Examples)

INNER PRODUCT SPACE	**INNER PRODUCT** $\langle \cdot, \cdot \rangle$
$\mathbb{R}^n$	$\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{k=1}^{n} u_k v_k = u_1 v_1 + \cdots + u_n v_n = \mathbf{u}^T \mathbf{v}$
$\mathbb{R}^{m \times n}$	$\langle A, B \rangle := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij} = a_{11} b_{11} + \cdots + a_{mn} b_{mn}$
P_n	$\langle p, q \rangle := \sum_{k=1}^{n+1} p(t_k) q(t_k) = p(t_1) q(t_1) + \cdots + p(t_{n+1}) q(t_{n+1})$
$C[a, b]$	$\langle f, g \rangle := \int_a^b f(x) g(x) \, dx$

REMARK: Other inner products are possible with these spaces but such inner products won't be considered in this course.

Inner Product (Properties)

Every inner product has the following basic properties by virtue of satisfying the inner product axioms:

Corollary

(Properties of Inner Products)

Let $(V, \langle\cdot,\cdot\rangle)$ *be an inner product space.*

Let "vectors" $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ *and scalar* $\alpha \in \mathbb{R}$. *Then:*

(*IP1*)	$\langle \mathbf{u}, \alpha\mathbf{v}\rangle = \alpha\langle \mathbf{u}, \mathbf{v}\rangle$	*Associativity of Inner Product*
(*IP2*)	$\langle \mathbf{u}, \vec{\mathbf{0}}\rangle = \langle \vec{\mathbf{0}}, \mathbf{u}\rangle = 0$	*Inner Product with* $\vec{\mathbf{0}}$ *is Zero Scalar*
(*IP3*)	$\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$	*Distributivity of Inner Product over VA*

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PROOF:

(IP1) $\langle \mathbf{u}, \alpha\mathbf{v}\rangle \overset{IPS3}{=} \langle \alpha\mathbf{v}, \mathbf{u}\rangle \overset{IPS4}{=} \alpha\langle \mathbf{v}, \mathbf{u}\rangle \overset{IPS3}{=} \alpha\langle \mathbf{u}, \mathbf{v}\rangle$

(IP2) $\langle \mathbf{u}, \vec{\mathbf{0}}\rangle \overset{CI\vec{\mathbf{0}}}{=} \langle \mathbf{u}, \mathbf{v}+(-\mathbf{v})\rangle \overset{IPS5}{=} \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, -\mathbf{v}\rangle \overset{IP1}{=} \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle = 0$

$\langle \vec{\mathbf{0}}, \mathbf{u}\rangle = \langle (0)\mathbf{v}, \mathbf{u}\rangle \overset{IPS4}{=} (0)\langle \mathbf{v}, \mathbf{u}\rangle = 0$

(IP3) $\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle \overset{IPS3}{=} \langle \mathbf{w}, \mathbf{u}+\mathbf{v}\rangle \overset{IPS5}{=} \langle \mathbf{w}, \mathbf{u}\rangle + \langle \mathbf{w}, \mathbf{v}\rangle \overset{IPS3}{=} \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$ QED

Norms Induced by Inner Products (Definition)

It turns out that the concept of a **norm** carries over to general vector spaces.

For inner product spaces, a norm can be naturally defined via inner product:

Definition

(Norm Induced by an Inner Product)

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. Let "vector" $\mathbf{u} \in V$.

The **norm** $||\cdot||$ **induced by the inner product** $\langle\cdot,\cdot\rangle$ **of** V is defined to be:

$$||\mathbf{u}|| := \sqrt{\langle\mathbf{u},\mathbf{u}\rangle}$$

Definition

(Normed Vector Space)

A vector space V with a norm $||\cdot||$ is called an **normed vector space**.
A compact notation for a normed vector space is: $(V, ||\cdot||)$

REMARK: Other norms are possible and show up in **Numerical Analysis** and certain applications, but only the induced norm will be considered here.

Norms Induced by Inner Products (Properties)

All norms (including induced norms) satisfy the following norm axioms:

Corollary

(Norm Axioms for a Normed Vector Space)

Let $(V, ||\cdot||)$ *be a normed vector space.*

Let "vector" $\mathbf{u}, \mathbf{v} \in V$ *and scalar* $\alpha \in \mathbb{R}$. *Then the following are satisfied:*

(*NM1*)	$\lVert\mathbf{u}\rVert \geq 0$	*Non-Negativity of Norm*
(*NM2*)	$\lVert\mathbf{u}\rVert = 0 \iff \mathbf{u} = \vec{\mathbf{0}}$	*Only* $\vec{\mathbf{0}}$ *has Norm Zero*
(*NM3*)	$\lVert\alpha\mathbf{u}\rVert = \lvert\alpha\rvert\lVert\mathbf{u}\rVert$	*Norm of Scalar Multiple*
(*NM4*)	$\lVert\mathbf{u}+\mathbf{v}\rVert \leq \lVert\mathbf{u}\rVert + \lVert\mathbf{v}\rVert$	*Triangle Inequality for Norm*

REMARK: In (NM3), the absolute value bars around α are needed since α may be negative but norms are always non-negative (NM1).

REMARK: Other norms are possible and show up in **Numerical Analysis** and certain applications, but only the norm induced by an inner product will be considered in this course.

Metrics Induced by Norms (Properties)

For vector spaces, the notion of "distance" is expressed in terms of norm:

Definition

(Metric Induced by a Norm)

Let $(V, ||\cdot||)$ be a normed vector space. Let "vector" $\mathbf{u} \in V$.

The **metric** $d(\cdot,\cdot)$ **induced by the norm** $||\cdot||$ **of** V is defined to be:

$$d(\mathbf{u}, \mathbf{v}) := ||\mathbf{u} - \mathbf{v}|| \qquad \text{(read "the metric between } \mathbf{u} \text{ \& } \mathbf{v}\text{")}$$

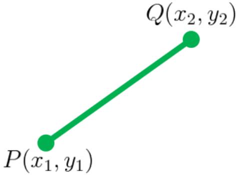
Corollary

(Metric Axioms for a Vector Space)

Let $(V, ||\cdot||)$ *be a normed vector space and* $d(\cdot,\cdot)$ *be the induced metric.*

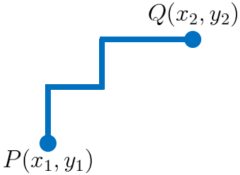
Let "vector" $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ *and scalar* $\alpha \in \mathbb{R}$*. Then the following are satisfied:*

(*MT1*)	$d(\mathbf{u}, \mathbf{v}) \geq 0$	*Non-Negativity of Metric*
(*MT2*)	$d(\mathbf{u}, \mathbf{v}) = 0 \iff \mathbf{u} = \mathbf{v}$	*Zero Metric means both "Vectors" are =*
(*MT3*)	$d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$	*Commutativity of Metric*
(*MT4*)	$d(\mathbf{u}, \mathbf{w}) \leq d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w})$	*Triangle Inequality for Metric*



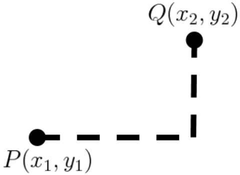
Euclidean Metric

$d(P, Q) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$



Taxicab Metric

$d(P, Q) := |x_1 - x_2| + |y_1 - y_2|$



Supremum Metric

$d(P, Q) := \max(|x_1 - x_2|, |y_1 - y_2|)$

Only **induced** metrics (e.g. Euclidean metric on $\mathbb{R}^2$) will be considered.

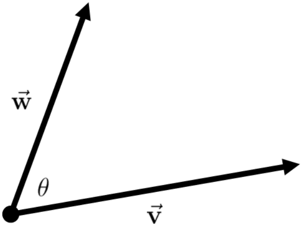
Inner Product Spaces (Standard Examples)

INNER PRODUCT SPACE	**INNER PRODUCT** $\langle \cdot, \cdot \rangle$	**INDUCED NORM** $\|\| \cdot \|\|$	**INDUCED METRIC** $d(\cdot, \cdot)$
$\mathbb{R}^n$	$\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{k=1}^{n} u_k v_k$	$\|\|\mathbf{u}\|\| := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$	$d(\mathbf{u}, \mathbf{v}) := \|\|\mathbf{u} - \mathbf{v}\|\|$
$\mathbb{R}^{m \times n}$	$\langle A, B \rangle := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$	$\|\|A\|\| := \sqrt{\langle A, A \rangle}$	$d(A, B) := \|\|A - B\|\|$
P_n	$\langle p, q \rangle := \sum_{k=1}^{n+1} p(t_k) q(t_k)$	$\|\|p\|\| := \sqrt{\langle p, p \rangle}$	$d(p, q) := \|\|p - q\|\|$
$C[a, b]$	$\langle f, g \rangle := \int_a^b f(x) g(x) \, dx$	$\|\|f\|\| := \sqrt{\langle f, f \rangle}$	$d(f, g) := \|\|f - g\|\|$

REMARK: The inner product, norm, metric of $\mathbb{R}^n$ shown above are sometimes called the **euclidean** inner product, norm, metric.

Inner Product (Coordinate-Free Definition)

Just as with dot product, inner products have coordinate-free definitions:



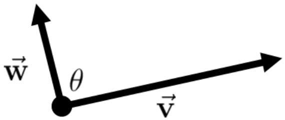
Definition

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space with induced norm $|| \cdot ||$.
Let "vectors" $\mathbf{v}, \mathbf{w} \in V$. Then:

$$\langle \mathbf{v}, \mathbf{w} \rangle = ||\mathbf{v}|| ||\mathbf{w}|| \cos\theta \qquad \text{where } \theta \in [0, \pi]$$

Inner Product (Orthogonality)

Orthogonality carries over to general inner product spaces:



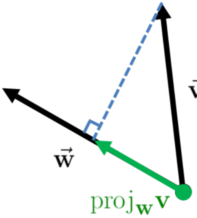
Theorem

Let $(V, \langle \cdot, \cdot \rangle)$ *be an inner product space. Let "vectors"* $\mathbf{v}, \mathbf{w} \in V$*. Then:*

"Vectors" $\mathbf{v}, \mathbf{w}$ *are* ***orthogonal*** $\iff \mathbf{v} \perp \mathbf{w} \iff \langle \mathbf{v}, \mathbf{w} \rangle = 0$

Orthogonal Projection onto a "Vector" (Formula)

Orthogonal projections carry over to general inner product spaces:



Definition

(Orthogonal Projection onto a "Vector")

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. Let "vectors" $\mathbf{v}, \mathbf{w} \in V$ s.t. $\mathbf{w} \neq \vec{\mathbf{0}}$.

Then the **(orthogonal) projection of $\mathbf{v}$ onto $\mathbf{w}$** is defined by:

$$\text{proj}_{\mathbf{w}}\mathbf{v} := \left(\frac{\langle \mathbf{v}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\right)\mathbf{w}$$

Fin

Fin.