Linear Transformations: Definition, Image

Linear Algebra

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TTU

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Common Vector Spaces

$$
\begin{aligned}
\mathbb{R} &\equiv \text{Set of all real numbers (scalars)} \\
\mathbb{R}^2 &\equiv \text{Set of all ordered pairs (2-wide vectors)} \\
\mathbb{R}^3 &\equiv \text{Set of all ordered triples (3-wide vectors)} \\
\mathbb{R}^n &\equiv \text{Set of all ordered } n\text{-tuples } (n\text{-wide vectors}) \\
\mathbb{R}^{m\times n} &\equiv \text{Set of all } m \times n \text{ matrices} \\
\mathbb{R}^{n\times n} &\equiv \text{Set of all } n \times n \text{ square matrices} \\
P_n &\equiv \text{Set of all polynomials of degree } n \text{ or less} \\
C[a,b] &\equiv \text{Set of all continuous functions on } [a,b] \\
C^1[a,b] &\equiv \text{Set of all differentiable functions on } [a,b] \\
C^2[a,b] &\equiv \text{Set of all twice-differentiable fcns on } [a,b] \\
C(\mathbb{R}) &\equiv \text{Set of all everywhere-continuous functions} \\
C^1(\mathbb{R}) &\equiv \text{Set of all everywhere-differentiable functions} \\
C^2(\mathbb{R}) &\equiv \text{Set of all everywhere-twice-differentiable fcns} \\
C^\infty(\mathbb{R}) &\equiv \text{Set of all everywhere-infinitely-differentiable fcns}
\end{aligned}
$$

REMARK: Always assume that the operations of vector addition & scalar multiplication are the standard definitions.

One could define these operations in other ways, but such scenarios are dealt with extensively in **Abstract Algebra**. (MATH 3360)

Transformation (Definition)

Definition

(Transformation)

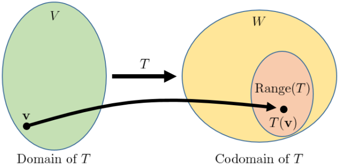
A **transformation** T is a function between vector spaces V, W.

The **signature** of transformation T from vector spaces V into W is written:

$$T : V \to W$$

V is the **domain** of T and W is the **codomain** of T.
$T(\mathbf{v}) \in W$ is called the **image** of vector $\mathbf{v} \in V$.
The set of all images of all vectors in V is the **range** of T:

$$\text{Range}(T) := \{T(\mathbf{v}) : \mathbf{v} \in V\} \subseteq W$$



Examples of Transformations

SIGNATURE	EXAMPLE	SPECIAL NAME
$T : \mathbb{R} \to \mathbb{R}$	$T(x) = x^2 + 1$	(Scalar) Function
$T : \mathbb{R} \to \mathbb{R}^n$	$T(t) = \begin{bmatrix} 1 - t \\ \sqrt{t} \end{bmatrix}$	Vector Function
$T : \mathbb{R}^m \to \mathbb{R}$	$T(\mathbf{x}) = 2\mathbf{x}^T\mathbf{x}$	Scalar Field
$T : \mathbb{R}^m \to \mathbb{R}$	$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1^3 - x_1x_2$	Scalar Field
$T : \mathbb{R}^m \to \mathbb{R}^n$	$T(\mathbf{x}) = A\mathbf{x} \qquad (A \in \mathbb{R}^{3\times 4})$	Vector Field
$T : \mathbb{R}^m \to \mathbb{R}^n$	$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \sin x_3 \\ \ln x_2 - x_1^2 \end{bmatrix}$	Vector Field

Examples of Transformations

SIGNATURE	EXAMPLE	SPECIAL NAME
$T : \mathbb{R} \to \mathbb{R}^{m \times n}$	$T(t) = t^2 I \qquad (I \text{ is } 3 \times 3 \text{ identity matrix})$	Matrix Function
$T : \mathbb{R} \to \mathbb{R}^{m \times n}$	$T(t) = \begin{bmatrix} 8 & (1 - t^4) \\ \cos(\pi t) & 5e^t \end{bmatrix}$	Matrix Function
$T : \mathbb{R}^{n \times n} \to \mathbb{R}$	$T(A) = \det(A) + 7$	????
$T : \mathbb{R}^{m \times n} \to \mathbb{R}$	$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right) = a_{11}a_{21} - a_{32}^2$	????
$T : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$	$T(A) = I - 3A$	????
$T : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$	$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right) = \begin{bmatrix} a_{23}^2 & 0 \\ a_{13}a_{11} & a_{13} \\ \sqrt[3]{a_{11}} & 6a_{22} \end{bmatrix}$	????

Examples of Transformations

SIGNATURE	EXAMPLE	SPECIAL NAME
$T : P_n \to \mathbb{R}$	$T(p) = 3p(1) - 2p(-1) + 4$	????
$T : P_2 \to \mathbb{R}$	$T(ax^2 + bx + c) = 3a - b^4 + c^{2/3}$	????
$T : P_{n+2} \to P_n$	$T(p) = p''(x)$	????
$T : P_n \to P_{n+1}$	$T(p) = \int p(x)\, dx$	????
$T : C^2(\mathbb{R}) \to \mathbb{R}$	$T(f) = 3f''(2) - 2f'(1) + f(0)$	Functional
$T : C[0,1] \to \mathbb{R}$	$T(f) = \int_0^1 x^2 f(x)\, dx$	Functional

Linear Transformation (Definition)

The most useful transformations are linear transformations:

Definition

(Linear Transformation)

Let V, W be vector spaces and transformation $T : V \to W$. Then:

T is a **linear transformation** if the following all hold $\forall \mathbf{u}, \mathbf{v} \in V$ and $\forall \alpha \in \mathbb{R}$:

(LT1)	$T(\mathbf{u}+\mathbf{v})$	$=$	$T(\mathbf{u})+T(\mathbf{v})$	Preservation of Vector Addition
(LT2)	$T(\alpha\mathbf{v})$	$=$	$\alpha T(\mathbf{v})$	Preservation of Scalar Mult.

i.e. Linear transformations preserve vector addition & scalar multiplication.

REMARK: If you are told a transformation is linear a priori, then the linear transformation will be denoted by L instead of T.

Linear Transformation (Properties)

Theorem

(Properties Linear Transformation)

Let $L : V \to W$ *be a linear transformation.*
Then the following all hold $\forall \mathbf{u}, \mathbf{v} \in V$ *and* $\forall \alpha, \beta \in \mathbb{R}$*:*

(*LT3*)	$L(\vec{\mathbf{0}})$	$=$	$\vec{\mathbf{0}}$	L *maps* $\vec{\mathbf{0}} \in V$ *to* $\vec{\mathbf{0}} \in W$
(*LT4*)	$L(\mathbf{u} - \mathbf{v})$	$=$	$L(\mathbf{u}) - L(\mathbf{v})$	*Preservation of Vector Subtraction*
(*LT5*)	$L(\alpha\mathbf{u} + \beta\mathbf{v})$	$=$	$\alpha L(\mathbf{u}) + \beta L(\mathbf{v})$	*Superposition Principle*

Establishing (LT5) is sufficient when showing T is a linear transformation.
(LT3) may be helpful when showing T is not a linear transformation.

Corollary

(General Superposition Principle)

Let $L : V \to W$ *be a linear transformation and* $\mathbf{v} \in V$*. Then:*

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \implies L(\mathbf{v}) = c_1L(\mathbf{v}_1) + c_2L(\mathbf{v}_2) + \cdots + c_nL(\mathbf{v}_n)$$

where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ *and* $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

Linear Transformation (Properties)

Theorem

(Properties Linear Transformation)

Let $L : V \to W$ be a linear transformation.
Then the following all hold $\forall \mathbf{u}, \mathbf{v} \in V$ and $\forall \alpha, \beta \in \mathbb{R}$:

(*LT3*)	$L(\vec{\mathbf{0}})$	$=$	$\vec{\mathbf{0}}$	*L maps $\vec{\mathbf{0}} \in V$ to $\vec{\mathbf{0}} \in W$*
(*LT4*)	$L(\mathbf{u} - \mathbf{v})$	$=$	$L(\mathbf{u}) - L(\mathbf{v})$	*Preservation of Vector Subtraction*
(*LT5*)	$L(\alpha\mathbf{u} + \beta\mathbf{v})$	$=$	$\alpha L(\mathbf{u}) + \beta L(\mathbf{v})$	*Superposition Principle*

PROOF:

$$L(\vec{\mathbf{0}}) = L((0)\mathbf{v}) \overset{LT2}{=} (0)L(\mathbf{v}) = \vec{\mathbf{0}}$$

$$L(\mathbf{u} - \mathbf{v}) = L(\mathbf{u} + (-1)\mathbf{v}) \overset{LT1}{=} L(\mathbf{u}) + L((-1)\mathbf{v}) \overset{LT2}{=} L(\mathbf{u}) + (-1)L(\mathbf{v}) = L(\mathbf{u}) - L(\mathbf{v})$$

$$L(\alpha\mathbf{u} + \beta\mathbf{v}) \overset{LT1}{=} L(\alpha\mathbf{u}) + L(\beta\mathbf{v}) \overset{LT2}{=} \alpha L(\mathbf{u}) + \beta L(\mathbf{v})$$

QED

Two Special Linear Transformations

Proposition

(Zero Transformation & Identity Transformation)

Let $L : V \to W$ be a linear transformation. Then:

(ZT)	*L is the*	***zero transformation***	*if*	$L(\mathbf{v}) = \vec{\mathbf{0}} \quad \forall \mathbf{v} \in V$
(IT)	*L is the*	***identity transformation***	*if*	$L(\mathbf{v}) = \vec{\mathbf{v}} \quad \forall \mathbf{v} \in V$

PROOF (that zero & identity transformations are linear):

Let T be the zero transformation (ZT). Then:

$$T(\alpha\mathbf{u} + \beta\mathbf{v}) \stackrel{ZT}{=} \vec{\mathbf{0}} = (\alpha)\vec{\mathbf{0}} + (\beta)\vec{\mathbf{0}} \stackrel{ZT}{=} \alpha T(\mathbf{u}) + \beta T(\mathbf{v}) \implies \text{ZT } \underline{\text{is}} \text{ linear}$$

Let T be the identity transformation (IT). Then:

$$T(\alpha\mathbf{u} + \beta\mathbf{v}) \stackrel{IT}{=} \alpha\mathbf{u} + \beta\mathbf{v} \stackrel{IT}{=} \alpha T(\mathbf{u}) + \beta T(\mathbf{v}) \implies \text{IT } \underline{\text{is}} \text{ linear}$$

QED

Linear Transformation given by a Matrix

Theorem

(Linear Transformation given by a Matrix)

Let A *be an* $m \times n$ *matrix.*

Then $T(\mathbf{v}) = A\mathbf{v}$ *is a linear transformation from* $\mathbb{R}^n$ *into* $\mathbb{R}^m$*:*

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

PROOF: The shape of matrix A forces the shapes of $\mathbf{v}$ & its image $T(\mathbf{v})$:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A\ (m\times n)} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{v}\ (n\times 1)} = \underbrace{\begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \qquad \vdots \qquad \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}}_{T(\mathbf{v})\ (m\times 1)}$$

$\implies$ $\mathbf{v} \in \mathbb{R}^n$ and $T(\mathbf{v}) \in \mathbb{R}^m$ $\implies$ $T : \mathbb{R}^n \to \mathbb{R}^m$

$$T(\alpha\mathbf{u} + \beta\mathbf{v}) = A(\alpha\mathbf{u} + \beta\mathbf{v}) = A(\alpha\mathbf{u}) + A(\beta\mathbf{v}) = \alpha(A\mathbf{u}) + \beta(A\mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v})$$

$\implies$ T is linear QED

Fin.