Linear Transformations: Kernel, Range, 1-1, Onto

Linear Algebra

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Kernel of a Linear Transformation (Definition)

Definition

(Kernel of a Linear Transformation)

Let $L : V \to W$ be a linear transformation.
Then the **kernel** of L is defined to be:

$$\text{ker}(L) := \{\mathbf{v} \in V : L(\mathbf{v}) = \vec{\mathbf{0}}\} \subseteq V$$

i.e. The kernel of L is the set of all vectors in V that are mapped by L to $\vec{\mathbf{0}}$.

Theorem

(The Kernel of L is a Subspace of V)

Let $L : V \to W$ be a linear transformation. Then, ker(L) is a subspace of V.

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Theorem

(The Kernel of L is a Subspace of V)

Let $L : V \to W$ be a linear transformation. Then, $\ker(L)$ is a subspace of V.

<u>PROOF:</u> $L(\vec{\mathbf{0}}) = \vec{\mathbf{0}} \implies \vec{\mathbf{0}} \in \ker(L) \implies \ker(L)$ is a nonempty subset of V.

$\mathbf{u}, \mathbf{v} \in \ker(L) \implies L(\mathbf{u} + \mathbf{v}) \overset{LT1}{=} L(\mathbf{u}) + L(\mathbf{v}) = \vec{\mathbf{0}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} \implies \mathbf{u} + \mathbf{v} \in \ker(L)$

$\alpha \in \mathbb{R},\ \mathbf{v} \in \ker(L) \implies L(\alpha\mathbf{v}) \overset{LT2}{=} \alpha L(\mathbf{v}) = \alpha(\vec{\mathbf{0}}) = \vec{\mathbf{0}} \implies \alpha\mathbf{v} \in \ker(L)$

$\therefore$ $\ker(L)$ is closed under VA & SM $\implies \ker(L)$ is a subspace of V QED

Range of a Linear Transformation (Definition)

Definition

(Range of a Linear Transformation)

Let $L : V \to W$ be a linear transformation.
Then the **range** of L is defined to be:

$$\text{range}(L) := \{L(\mathbf{v}) \in W : \mathbf{v} \in V\}$$

Theorem

(The Range of L is a Subspace of W)

Let $L : V \to W$ be a linear transformation. Then, range(L) is a subspace of W.

PROOF: $L(\vec{\mathbf{0}}) = \vec{\mathbf{0}} \implies \vec{\mathbf{0}} \in \text{range}(L) \implies \text{range}(L)$ is nonempty subset of W.

$L(\mathbf{u}), L(\mathbf{v}) \in \text{range}(L) \implies \mathbf{u}, \mathbf{v} \in V \implies (\mathbf{u} + \mathbf{v}) \in V \implies L(\mathbf{u} + \mathbf{v}) \in \text{range}(L)$

$\alpha \in \mathbb{R},\ L(\mathbf{v}) \in \text{range}(L) \implies \mathbf{v} \in V \implies \alpha\mathbf{v} \in V \implies L(\alpha\mathbf{v}) \in \text{range}(L)$

$\therefore$ range(L) is closed under VA & SM $\implies$ range(L) is a subspace of W $\square$

Ker(L) & Range(L) in terms of Representative Matrix

Corollary

(Relationship between a Linear Transformation and its Representative Matrix)

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ *such that* $L(\mathbf{x}) = A\mathbf{x}$. *(where* $A \in \mathbb{R}^{m \times n}$*)* *Then:*

$$\textit{ker}(L) = \textit{NulSp}(A) \qquad \textit{range}(L) = \textit{ColSp}(A)$$

<u>PROOF:</u>

$\mathbf{x} \in \text{ker}(L) \subseteq \mathbb{R}^n \iff L(\mathbf{x}) = \vec{\mathbf{0}} \iff A\mathbf{x} = \vec{\mathbf{0}} \iff \mathbf{x} \in \text{NulSp}(A)$

$L(\mathbf{x}) \in \text{range}(L) \subseteq \mathbb{R}^m \iff \mathbf{x} \in \mathbb{R}^n \iff A\mathbf{x} \in \text{ColSp}(A)$

Rank & Nullity of a Linear Transformation

Definition

(Rank & Nullity of a Matrix & Linear Transformation)

(i) Let A be an $m \times n$ matrix. Then:

$$\begin{aligned} \text{rank}(A) &= \dim[\text{ColSp}(A)] = (\text{\# of pivot columns of RREF}(A)) \\ \text{nullity}(A) &= \dim[\text{NulSp}(A)] = (\text{\# of non-pivot columns of RREF}(A)) \end{aligned}$$

(ii) Let $L : V \to W$ be a linear transformation. Then:

$$\text{rank}(L) = \dim[\text{range}(L)] \qquad \text{nullity}(L) = \dim[\ker(L)]$$

Rank-Nullity Theorem

Theorem

(Rank-Nullity Theorem)

(i) Let A be an $m \times n$ matrix. Then:

$$\textit{rank}(A) + \textit{nullity}(A) = (\textit{\# of columns of } A)$$

(ii) Let $L : V \to W$ be a linear transformation. Then:

$$\textit{rank}(L) + \textit{nullity}(L) = \textit{dim}[\textit{domain}(L)]$$

Preimage of a Vector (Definition)

Definition

(Preimage of a Vector)

Let $T : V \rightarrow W$ be a transformation. Then the **preimage** of vector $\mathbf{w} \in W$ is:

$$\text{Preimage}(\mathbf{w}) := \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{w}\} \subseteq V$$

1-1 Linear Transformations (Definition)

Definition

(1-1 Transformation)

Let $T : V \to W$ be a transformation. Then:

$$T \text{ is } \textbf{1-1} \ \textbf{(one-to-one)} \iff \forall \mathbf{u}, \mathbf{v} \in V, \ \ T(\mathbf{u}) = T(\mathbf{v}) \implies \mathbf{u} = \mathbf{v}.$$

Definition

(1-1 Linear Transformation)

Let $L : V \to W$ be a linear transformation. Then, L is 1-1 $\iff$ $\text{ker}(L) = \{\vec{\mathbf{0}}\}$.

Onto Linear Transformations (Definition)

Definition

(Onto Transformation)

Let $T : V \to W$ be a transformation. Then, T is **onto** $\iff$ range$(T) = W$.

Definition

(Onto Linear Transformation)

Let $L : V \to W$ be a linear transformation such that W is finite-dimensional. Then, L is onto $\iff$ rank$(L) =$ dim(W).

Isomorphisms (Definition)

Definition

(Isomorphism)

Let $L : V \to W$ be a linear transformation.
Then, L is called an **isomorphism** if L is 1-1 and onto.

Moreover, vector spaces V, W are said to be **isomorphic** (to each other) if there exists an isomorphism from V to W.

Theorem

(Finite-dimensional Isomorphic Spaces have the same Dimension)

Let V, W be finite-dimensional vector spaces.
Then, V & W are isomorphic $\iff$ dim$(V) = $ dim(W)

PROOF: See the textbook if interested.

Isomorphisms (Examples)

Vector spaces $\mathbb{R}^4$, $\mathbb{R}^{2\times 2}$, P_3 are all isomorphic to each other since there exists isomorphisms L_1, L_2 as shown below:

Let $L_1 : \mathbb{R}^{2\times 2} \to \mathbb{R}^4$ s.t. $L_1\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$

Let $L_2 : P_3 \to \mathbb{R}^4$ s.t. $L_2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$

Isomorphic vector spaces are essentially ”the same” but with representations by different mathematical objects (vectors, matrices, polynomials.)

In particular, matrices & polynomials can be represented by vectors, which will be quite useful in the next two sections....

Fin

Fin.