Linear Transformations: Standard Matrix

Linear Algebra

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TTU

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PART I:

STANDARD MATRIX (THE EASY CASE)
COMPOSITION OF LINEAR TRANSFORMATIONS
INVERSE OF A LINEAR TRANSFORMATION

Representing a Linear Transformation by a Matrix

It's desirable to represent linear transformations as matrix-vector products:

$$L(\mathbf{x}) = A\mathbf{x}$$

But how to systematically achieve this??

Consider linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$L(x_1, x_2) = (4x_2 - x_1, x_1 + x_2, 3x_2) \iff L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_2 - x_1 \\ x_1 + x_2 \\ 3x_2 \end{bmatrix}$$

Representing a Linear Transformation by a Matrix

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$$L(x_1, x_2) = (4x_2 - x_1, x_1 + x_2, 3x_2) \iff L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_2 - x_1 \\ x_1 + x_2 \\ 3x_2 \end{bmatrix}$$

$$L(\mathbf{x}) = L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_2 - x_1 \\ x_1 + x_2 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_2 \\ x_2 \\ 3x_2 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}$$

However, recall the **standard basis** for $\mathbb{R}^2$: $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \equiv \{\mathbf{e}_1, \mathbf{e}_2\}$

Then,

$$L(\mathbf{e}_1) = A\mathbf{e}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, L(\mathbf{e}_2) = A\mathbf{e}_2 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \implies A = \begin{bmatrix} | & | \\ L(\mathbf{e}_1) & L(\mathbf{e}_2) \\ | & | \end{bmatrix}$$

The Standard Matrix for a Linear Transformation

Definition

(Standard Matrix for a Linear Transformation)

Let linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ s.t. $L(\mathbf{x}) = A\mathbf{x} \;\; \forall \mathbf{x} \in \mathbb{R}^n$, where $A \in \mathbb{R}^{m \times n}$.
Then A is called the **standard matrix** for linear transformation L.

Proposition

(Finding the Standard Matrix – Easy Case)

GIVEN: Linear Transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ s.t. L is explicitly defined.

TASK: Find Standard Matrix $A \in \mathbb{R}^{m \times n}$ s.t. $L(\mathbf{x}) = A\mathbf{x}$

(1) $A = \begin{bmatrix} | & | & & | \\ L(\mathbf{e}_1) & L(\mathbf{e}_2) & \cdots & L(\mathbf{e}_n) \\ | & | & & | \end{bmatrix}$

*i.e. The columns of A are the images of the **standard basis vectors** for $\mathbb{R}^n$.*

Composition of Linear Transformations

Theorem

(Composition of Linear Transformations)

Let linear transformation $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ *s.t.* $L_1(\mathbf{x}) = A_1\mathbf{x}$.
Let linear transformation $L_2 : \mathbb{R}^m \to \mathbb{R}^p$ *s.t.* $L_2(\mathbf{x}) = A_2\mathbf{x}$.

Then the ***composition*** $L_2 \circ L_1$ *is defined by* $(L_2 \circ L_1)(\mathbf{x}) := L_2[L_1(\mathbf{x})]$.

Moreover, the composition $L_2 \circ L_1$ *is a linear transformation and*

$$(L_2 \circ L_1)(\mathbf{x}) = (A_2A_1)\mathbf{x}$$

PROOF:

$$\begin{aligned}(L_2 \circ L_1)(\mathbf{x}+\mathbf{y}) &= L_2[L_1(\mathbf{x}+\mathbf{y})] \overset{LT1}{=} L_2[L_1(\mathbf{x}) + L_1(\mathbf{y})] \overset{LT1}{=} L_2[L_1(\mathbf{x})] + L_2[L_1(\mathbf{y})] \\ &= (L_2 \circ L_1)(\mathbf{x}) + (L_2 \circ L_1)(\mathbf{y})\end{aligned}$$

$$(L_2 \circ L_1)(\alpha\mathbf{x}) = L_2[L_1(\alpha\mathbf{x})] \overset{LT2}{=} L_2[\alpha L_1(\mathbf{x})] \overset{LT2}{=} \alpha L_2[L_1(\mathbf{x})] = \alpha(L_2 \circ L_1)(\mathbf{x})$$

$\therefore$ $L_2 \circ L_1$ is a linear transformation.

$$(L_2 \circ L_1)(\mathbf{x}) = L_2[L_1(\mathbf{x})] = L_2(A_1\mathbf{x}) = A_2(A_1\mathbf{x}) \overset{M1}{=} (A_2A_1)\mathbf{x}$$

QED

Inverse Linear Transformation

Definition

(Inverse Linear Transformation)

Let linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ have identical domain & codomain.

Then linear transformation $L^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is the **inverse** of L if

$$(L^{-1} \circ L)(\mathbf{x}) = \mathbf{x} \ \text{ and } \ (L \circ L^{-1})(\mathbf{x}) = \mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n$$

L is called **invertible** if its inverse L^{-1} exists.

Theorem

(Inverse Linear Transformation in terms of Standard Matrix)

Let linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ *s.t.* $L(\mathbf{x}) = A\mathbf{x}$.

If L *is invertible, then its inverse* $L^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $L^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$

PART II:

STANDARD MATRIX (THE HARD CASE)
STANDARD MATRIX FOR DERIVATIVE OF POLYNOMIALS
STANDARD MATRIX FOR INTEGRAL OF POLYNOMIALS

Representing a Linear Transformation by a Matrix

As seen in PART I, it's straightforward to find the standard matrix A for a linear transformation L when:

(EASY CASE) L is defined explicitly in terms of components of $\mathbf{x}$:

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_2 - x_1 \\ x_1 + x_2 \\ 3x_2 \end{bmatrix}$$

——— OR ———

(EASY CASE) L is described in terms of images of standard basis vectors:

$$L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -7 \\ -2 \end{bmatrix}, \text{ and } L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$$

——— BUT HOW TO HANDLE ———

(HARD CASE) L is described in terms of images of non-std basis vectors:

$$L\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -7 \\ -2 \end{bmatrix}, \text{ and } L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$$

Standard Matrix – The Hard Case (Motivation)

Given $L\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -7 \\ -2 \end{bmatrix}$, and $L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$, find A s.t. $L(\mathbf{x}) = A\mathbf{x}$

$$\Longrightarrow \begin{cases} L\left((2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -7 \\ -2 \end{bmatrix} \\ L\left((1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (2)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix} \end{cases}$$

$$\Longrightarrow \begin{cases} (2)L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + (-1)L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -7 \\ -2 \end{bmatrix} \\ (1)L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + (2)L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix} \end{cases} \qquad \left(\begin{array}{c} \textit{Superposition} \\ \textit{Principle} \end{array}\right)$$

$$\Longrightarrow \left[\begin{array}{cc|c} 2 & -1 & (0,-7,-2)^T \\ 1 & 2 & (5,4,4)^T \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & 0 & (1,-2,0)^T \\ 0 & \boxed{1} & (2,3,2)^T \end{array}\right]$$

Standard Matrix – The Hard Case (Motivation)

Given $L\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -7 \\ -2 \end{bmatrix}$, and $L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$, find A s.t. $L(\mathbf{x}) = A\mathbf{x}$

$$\Longrightarrow \left[\begin{array}{cc|c} 2 & -1 & (0,-7,-2)^T \\ 1 & 2 & (5,4,4)^T \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & 0 & (1,-2,0)^T \\ 0 & \boxed{1} & (2,3,2)^T \end{array}\right]$$

$$\Longrightarrow L(\mathbf{e}_1) = L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \text{ and } L(\mathbf{e}_2) = L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

$$\therefore\ A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \\ 0 & 2 \end{bmatrix} \Longrightarrow L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Finding the Standard Matrix – Hard Case

Proposition

(Finding the Standard Matrix – Hard Case)

<u>*GIVEN:*</u> *Linear Transformation* L *s.t. images* $L(\mathbf{b}_1), \ldots, L(\mathbf{b}_n)$ *are provided.*

where $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ *is a non-standard basis for* $\mathbb{R}^n$*.*

<u>*TASK:*</u> *Find Standard Matrix* $A \in \mathbb{R}^{m \times n}$ *s.t.* $L(\mathbf{x}) = A\mathbf{x}$

(1) $$\left[\begin{array}{c|c} \text{——} \quad \mathbf{b}_1 \quad \text{——} & L(\mathbf{b}_1) \\ \vdots & \vdots \\ \text{——} \quad \mathbf{b}_n \quad \text{——} & L(\mathbf{b}_n) \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{c|c} & L(\mathbf{e}_1) \\ I & \vdots \\ & L(\mathbf{e}_n) \end{array}\right]$$

<u>*NOTE:*</u> *Write non-standard basis vectors* $\mathbf{b}_1, \ldots, \mathbf{b}_n$ *as* ***row vectors****.*
<u>*NOTE:*</u> *Write the images* $L(\mathbf{b}_1), \ldots, L(\mathbf{b}_n)$ *as* ***transposes*** *to avoid confusion.*

(2) $$A = \begin{bmatrix} | & & | \\ L(\mathbf{e}_1) & \cdots & L(\mathbf{e}_n) \\ | & & | \end{bmatrix}$$

Isomorphisms (Review)

Vector spaces $\mathbb{R}^4,\ \mathbb{R}^{2\times 2},\ P_3$ are all isomorphic to each other since there exists isomorphisms L_1, L_2 as shown below:

Let $L_1 : \mathbb{R}^{2\times 2} \to \mathbb{R}^4$ s.t. $L_1\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$

Let $L_2 : P_3 \to \mathbb{R}^4$ s.t. $L_2\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$

Isomorphic vector spaces are essentially "the same" but with representations by different mathematical objects (vectors, matrices, polynomials.)

In particular, matrices & polynomials can be represented by vectors, which will be quite useful now....

<u>NOTATION:</u> $P_3 \simeq \mathbb{R}^4$ means 'P_3 is isomorphic to $\mathbb{R}^4$'

Linear Transformations involving Calculus

Differentiation & integration are in fact linear transformations:

$$\frac{d}{dx}\Big[f(x)+g(x)\Big] = f'(x)+g'(x) \qquad \frac{d}{dx}\Big[\alpha f(x)\Big] = \alpha f'(x)$$

$$\int \Big[f(x)+g(x)\Big]\,dx = \int f(x)\,dx + \int g(x)\,dx \qquad \int \Big[\alpha f(x)\Big]\,dx = \alpha \int f(x)\,dx$$

$$\int_a^b \Big[f(x)+g(x)\Big]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \qquad \int_a^b \Big[\alpha f(x)\Big]\,dx = \alpha \int_a^b f(x)\,dx$$

If one restricts attention to a vector space of polynomials, P_n, this means that differentation & integration can be represented by standard matrices!!

QUESTION: How to find the standard matrix for calculus of polynomials?
ANSWER: Recognize that P_n is isomorphic to $\mathbb{R}^{n+1}$.

WEX 6-3-1: Find standard matrix A for linear transformation $L : P_2 \to P_1$ s.t.

$$L(p) = p'(t)$$

Linear Transformations involving Calculus

WEX 6-3-1: Find standard matrix A for linear transformation $L : P_2 \to P_1$ s.t.

$$L(p) = p'(t)$$

Find derivative of arbitrary polynomial in P_2: $\frac{d}{dt}\left[a + bt + ct^2\right] = b + 2ct$

Apply isomorphisms: $a + bt + ct^2 \simeq \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $b + 2ct \simeq \begin{bmatrix} b \\ 2c \end{bmatrix}$

Then: $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \end{bmatrix} \implies A$ is $2 \times 3 \implies \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \end{bmatrix}$

$$\therefore \quad \boxed{A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}} = \begin{bmatrix} | & | & | \\ L(\mathbf{e}_1) & L(\mathbf{e}_2) & L(\mathbf{e}_3) \\ | & | & | \end{bmatrix}$$ where $\begin{aligned} \mathbf{e}_1 &= 1 \\ \mathbf{e}_2 &= t \\ \mathbf{e}_3 &= t^2 \end{aligned}$

$$L(\mathbf{e}_1) = 0 \simeq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, L(\mathbf{e}_2) = \frac{d}{dt}\left[t\right] = 1 \simeq \begin{bmatrix} 1 \\ 0 \end{bmatrix}, L(\mathbf{e}_3) = \frac{d}{dt}\left[t^2\right] = 2t \simeq \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Finding the Standard Matrix for Calculus (Procedure)

Proposition

(Finding the Standard Matrix for Linear Transformations involving Calculus)

GIVEN: *Linear Transformation* $L : P_n \to P_m$ *s.t.* L *involves calculus.*

TASK: *Find Standard Matrix* A *s.t.* $L(\mathbf{x}) = A\mathbf{x}$

(1) Compute images $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_{n+1})$.

(2) Produce the isomorphisms of these images in P_m *to images in* $\mathbb{R}^{m+1}$*:*

$$[L(\mathbf{e}_1)]_{\mathcal{E}'}, [L(\mathbf{e}_2)]_{\mathcal{E}'}, \ldots, [L(\mathbf{e}_{n+1})]_{\mathcal{E}'}$$

(3) $A = \begin{bmatrix} | & | & & | \\ [L(\mathbf{e}_1)]_{\mathcal{E}'} & [L(\mathbf{e}_2)]_{\mathcal{E}'} & \cdots & [L(\mathbf{e}_{n+1})]_{\mathcal{E}'} \\ | & | & & | \end{bmatrix}$

$\mathcal{E} = \{1, t, t^2, \ldots, t^n\} \equiv \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n+1}\}$ *is standard ordered basis for* P_n.

$\mathcal{E}' = \{1, t, t^2, \ldots, t^m\} \equiv \{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_{m+1}\}$ *is standard ordered basis for* P_m.

IMPORTANT: ***Always write polynomials in ascending order:*** $3 - t + 5t^2$

The Value of the Standard Matrix for Calculus

By knowing the standard matrix for the derivative of a quadratic polynomial...

$$\frac{d}{dt}\left[a+bt+ct^2\right] \simeq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \end{bmatrix} \simeq b+2ct$$

....derivatives of quadratics can be found simply with a matrix multiplication:

$$\frac{d}{dt}\left[t^2-t\right] \simeq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \simeq 2t-1$$

$$\frac{d}{dt}\left[3t-7\right] \simeq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -7 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \simeq 3$$

$$\frac{d}{dt}\left[2-4t^2\right] \simeq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \end{bmatrix} \simeq -8t$$

$$\frac{d}{dt}\left[5\right] \simeq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \simeq 0$$

Fin.