Sqaure Matrices: Eigenvalues, Eigenvectors

Linear Algebra

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TTU

20 November 2015

PART I:

EIGENVALUES, EIGENVECTORS, EIGENSPACES

CASE I: DISTINCT REAL EIGENVALUES

When a Matrix $\times$ Vector effectively Scales the Vector

Consider the following linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$L(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad A = \begin{bmatrix} -11 & -7 \\ 14 & 10 \end{bmatrix}$$

Then, in particular, if $\mathbf{x}_1 = (-1, 1)^T, \mathbf{x}_2 = (-1, 2)^T, \mathbf{x}_3 = (-1, -1)^T$:

$$L(\mathbf{x}_1) = A\mathbf{x}_1 = \begin{bmatrix} -11 & -7 \\ 14 & 10 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = (-4) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -4\mathbf{x}_1$$

—- AND —-

$$L(\mathbf{x}_2) = A\mathbf{x}_2 = \begin{bmatrix} -11 & -7 \\ 14 & 10 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = (3) \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 3\mathbf{x}_2$$

—- BUT —-

$$L(\mathbf{x}_3) = A\mathbf{x}_3 = \begin{bmatrix} -11 & -7 \\ 14 & 10 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 18 \\ -24 \end{bmatrix} \neq (\alpha) \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \alpha\mathbf{x}_3$$

i.e. The matrix-vector product sometimes reduces to a scalar-vector product!!!

But such behavior does not occur to just any vector one chooses!

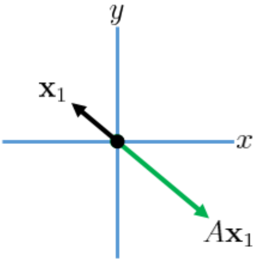
When a Matrix × Vector effectively Scales the Vector

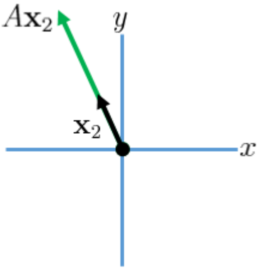
Consider the following linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that

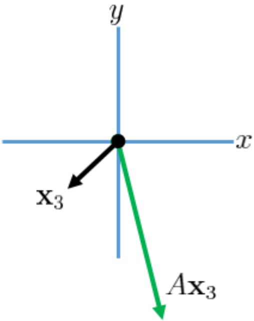
$$L(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad A = \begin{bmatrix} -11 & -7 \\ 14 & 10 \end{bmatrix}$$

Then, in particular, if $\mathbf{x}_1 = (-1, 1)^T, \mathbf{x}_2 = (-1, 2)^T, \mathbf{x}_3 = (-1, -1)^T$:

$L(\mathbf{x}_1) = A\mathbf{x}_1 = -4\mathbf{x}_1$ $\quad$ $L(\mathbf{x}_2) = A\mathbf{x}_2 = 3\mathbf{x}_2$ $\quad$ $L(\mathbf{x}_3) = A\mathbf{x}_3 \neq \alpha\mathbf{x}_3$







Eigenvalues & Eigenvectors of a Square Matrix (Def'n)

This "Matrix-Vector product reducing to Scalar-Vector product" behavior occurs with vectors called **eigenvectors**:

Definition

(Eigenvalues & Eigenvectors of a Square Matrix)

Let square matrix $A \in \mathbb{R}^{n \times n}$, non-zero vector $\mathbf{x} \in \mathbb{R}^n$, and scalar $\lambda \in \mathbb{R}$.

Then λ is an **eigenvalue** of A & $\mathbf{x}$ is a corresponding **eigenvector** of A if

$$\text{(EIG)} \qquad A\mathbf{x} = \lambda\mathbf{x} \qquad (\text{where } \mathbf{x} \neq \vec{\mathbf{0}})$$

Moreover, the ordered pair $(\lambda, \mathbf{x})$ is called an **eigenpair** of A.

"eigen" is pronounced EYE-gen.

"eigen" comes from German:

"der Eigenwert"	means	"own value"
"der Eigenvektor"	means	"own vector"

NOTE: Eigenvector $\mathbf{x} \neq \vec{\mathbf{0}}$ since $A\vec{\mathbf{0}} = \lambda\vec{\mathbf{0}}$ is true for all scalars $\lambda \in \mathbb{R}$

NOTE: It's possible to have **complex** eigenpairs (involving $i := \sqrt{-1}$), however only **real** eigenpairs will be considered in this course.

More Regarding Eigenvectors

Corollary

Let square matrix $A \in \mathbb{R}^{n \times n}$*. Then:*

(i) A scalar multiple of an eigenvector is also an eigenvector:

(EIG1) $(\lambda, \mathbf{x})$ *is an eigenpair of* $A \implies (\lambda, \alpha\mathbf{x})$ *is an eigenpair of* $A \quad (\alpha \neq 0)$

(ii) The sum of two eigenvectors with same eigenvalue is also an eigenvector:

(EIG2) $(\lambda, \mathbf{x}_1), (\lambda, \mathbf{x}_2)$ *are eigenpairs of* $A \implies (\lambda, \mathbf{x}_1 + \mathbf{x}_2)$ *is an eigenpair of* A

PROOF:

(i) Let $(\lambda, \mathbf{x})$ be an eigenpair of A. Then $A\mathbf{x} = \lambda\mathbf{x}$. Let $\alpha \neq 0$.

$$\implies A(\alpha\mathbf{x}) \overset{M2}{=} \alpha(A\mathbf{x}) \overset{EIG}{=} \alpha(\lambda\mathbf{x}) \overset{M2}{=} \lambda(\alpha\mathbf{x})$$
$$\implies A(\alpha\mathbf{x}) = \lambda(\alpha\mathbf{x})$$
$$\implies (\lambda, \alpha\mathbf{x}) \text{ is an eigenpair of } A$$

QED

More Regarding Eigenvectors

Corollary

Let square matrix $A \in \mathbb{R}^{n\times n}$. Then:

(i) A scalar multiple of an eigenvector is also an eigenvector:

(EIG1) $(\lambda, \mathbf{x})$ *is an eigenpair of* $A \implies (\lambda, \alpha\mathbf{x})$ *is an eigenpair of* $A \quad (\alpha \neq 0)$

(ii) The sum of two eigenvectors with same eigenvalue is also an eigenvector:

(EIG2) $(\lambda, \mathbf{x}_1), (\lambda, \mathbf{x}_2)$ *are eigenpairs of* $A \implies (\lambda, \mathbf{x}_1 + \mathbf{x}_2)$ *is an eigenpair of* A

PROOF:

(ii) Let $(\lambda, \mathbf{x}_1), (\lambda, \mathbf{x}_2)$ be eigenpairs of A. Then $A\mathbf{x}_1 = \lambda\mathbf{x}_1$ & $A\mathbf{x}_2 = \lambda\mathbf{x}_2$

$$\implies A(\mathbf{x}_1 + \mathbf{x}_2) \overset{M3}{=} A\mathbf{x}_1 + A\mathbf{x}_2 \overset{EIG}{=} \lambda\mathbf{x}_1 + \lambda\mathbf{x}_2 \overset{A7}{=} \lambda(\mathbf{x}_1 + \mathbf{x}_2)$$

$$\implies A(\mathbf{x}_1 + \mathbf{x}_2) = \lambda(\mathbf{x}_1 + \mathbf{x}_2)$$

$\implies (\lambda, \mathbf{x}_1 + \mathbf{x}_2)$ is an eigenpair of A

QED

Eigenspaces of a Square Matrix (Definition)

The previous corollary suggests that the set of all eigenvectors form a subspace provided the zero vector is also included in the set:

Definition

(Eigenspaces of a Square Matrix)

Let square matrix $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{R}$ be an eigenvalue of A.

Then the λ-**eigenspace** of A is the following subspace of $\mathbb{R}^n$:

$$E_\lambda := \{\mathbf{x} \in \mathbb{R}^n : (\lambda, \mathbf{x}) \text{ is an eigenpair of } A\} \cup \{\vec{\mathbf{0}}\}$$

i.e. The λ-eigenspace is the set of all eigenvectors of A with eigenvalue λ together with the zero vector (but of course $\vec{\mathbf{0}}$ is not an eigenvector.)

Finding Eigenvalues, Eigenvectors, Eigenspaces (Motivation)

So, how to find the eigenvalues, eigenvectors, eigenspaces of a matrix??

Let $(\lambda, \mathbf{x})$ be an eigenpair of square matrix A.

Then $A\mathbf{x} = \lambda\mathbf{x} \iff A\mathbf{x} - \lambda\mathbf{x} = \vec{\mathbf{0}} \iff A\mathbf{x} - \lambda I\mathbf{x} = \vec{\mathbf{0}} \overset{M4}{\iff} (A - \lambda I)\mathbf{x} = \vec{\mathbf{0}}$
where I is the $n \times n$ identity matrix.

Now, $(A - \lambda I)\mathbf{x} = \vec{\mathbf{0}}$ is a $n \times n$ homogeneous linear system,
which recall means that it automatically has the trivial solution $\mathbf{x} = \vec{\mathbf{0}}$.

However, since $\mathbf{x} = \vec{\mathbf{0}}$ is not an eigenvector of A,
the linear system $(A - \lambda I)\mathbf{x} = \vec{\mathbf{0}}$ must have non-trivial solns.

In order for $(A - \lambda I)\mathbf{x} = \vec{\mathbf{0}}$ to have non-trivial solutions,
the $n \times n$ matrix $(A - \lambda I)$ must not be invertible $\iff$ $\det(A - \lambda I) = 0$

$\therefore$ $(\lambda, \mathbf{x})$ is an eigenpair of square matrix $A \iff \det(A - \lambda I) = 0$

Characteristic Polynomial of a Square Matrix

$(\lambda, \mathbf{x})$ is an eigenpair of square matrix $A \iff \det(A - \lambda I) = 0$

Now, the unknown in the equation $\det(A - \lambda I) = 0$ is λ.

Recall that computing a determinant involves only additions & multiplications.

So, the expression $\det(A - \lambda I)$ is a polynomial in λ (and has a special name):

Definition

(Characteristic Polynomial of a Square Matrix)

Let square matrix $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{R}$ be an eigenvalue of A.
Then the **characteristic polynomial** of A is defined to be:

$$p_A(\lambda) := \det(\lambda I - A) = (-1)^n \det(A - \lambda I)$$

Moreover, $p_A(\lambda)$ is a polynomial in λ of degree n.
Moreover, the equation $p_A(\lambda) = 0$ is called the **characteristic equation** for A.

REMARK: Use $\det(\lambda I - A)$ for proofs & $\det(A - \lambda I)$ for computations.

NOTE: For $n \geq 3$, unless A is sparse, $p_A(\lambda)$ will be provided & factored a priori.

CASE I: Distinct Real Eigenvalues

Theorem

(Eigenvalues, Eigenvectors, and the Characteristic Polynomial)

Let square matrix $A \in \mathbb{R}^{n \times n}$*, non-zero vector* $\mathbf{x} \in \mathbb{R}^n$*, scalar* $\lambda \in \mathbb{R}$*. Then:*

(i) λ *is an eigenvalue of* $A \iff p_A(\lambda) = 0 \iff \mathit{det}(A - \lambda I) = 0$

(ii) $\mathbf{x}$ *is an eigenvector of* $A \iff (\lambda I - A)\mathbf{x} = \vec{\mathbf{0}} \iff (A - \lambda I)\mathbf{x} = \vec{\mathbf{0}}$

It's possible for some eigenvalues to be equal and have several linearly independent corresponding eigenvectors.

For now, let's consider the simpler case where all eigenvalues are distinct:

Theorem

(Distinct Real Eigenvalues)

Let square matrix $A \in \mathbb{R}^{n \times n}$ *s.t. all eigenvalues are real & distinct. Then:*

A *has* n *eigenpairs* $(\lambda_1, \mathbf{x}_1), (\lambda_2, \mathbf{x}_2), \cdots, (\lambda_n, \mathbf{x}_n)$ *s.t.* $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.

i.e. Distinct eigenvalue λ_k *has one distinct eigenvector* $\mathbf{x}_k$ *s.t.* $A\mathbf{x}_k = \lambda_k \mathbf{x}_k$.

CASE I: Distinct Real Eigenvalues (Procedure)

Proposition

(Finding Eigenvalues, Eigenvectors, Eigenspaces – Distinct Real Eigenvalues)

GIVEN: *Square Matrix* $A \in \mathbb{R}^{n \times n}$ *s.t. all eigenvalues are real & distinct.*

TASK: *Find the Eigenvalues* λ_k, *Eigenvectors* $\mathbf{x}_k$, *Eigenspaces* E_{λ_k} *of* A.

(1) Find Characteristic Polynomial $p_A(\lambda) = (-1)^n \text{det}(A - \lambda I)$

(2) Solve Characteristic Eqn $\text{det}(A - \lambda I) = 0$ *to find Eigenvalues* $\lambda_1, \ldots, \lambda_n$

(3) Find the Eigenspace for each Eigenvalue λ_k: $E_{\lambda_k} = \text{NullSp}(A - \lambda_k I)$

(4) Find an Eigenvector for each Eigenvalue λ_k: $\mathbf{x}_k = (\text{basis vector for } E_{\lambda_k})$

SANITY CHECKS: $A\mathbf{x}_k = \lambda_k \mathbf{x}_k$, $\text{dim}(E_{\lambda_k}) = 1$, $\mathbf{x}_k$*'s are distinct and non-zero*

$$p_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

REMARK: It's convention for eigenvalues to be indexed in increasing order:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < \lambda_n.$$

Eigenvalues of Diagonal & Triangular Matrices

It turns out the eigenvalues of a triangular matrix are immediate:

Proposition

(Eigenvalues of a Triangular Matrix)

The eigenvalues of a triangular matrix are the main diagonal entries.

Recall that a diagonal matrix is a special triangular matrix:

Proposition

(Eigenvalues of a Diagonal Matrix)

The eigenvalues of a diagonal matrix are the main diagonal entries.

WEX 7-1-1: Find the eigenvalues $\lambda_1 < \lambda_2$ of the matrices

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 8 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} 7 & 0 \\ 0 & 5 \end{bmatrix}$$

Matrix A: $\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 3 \end{array}$ Matrix B: $\begin{array}{l} \lambda_1 = -6 \\ \lambda_2 = -1 \end{array}$ Matrix C: $\begin{array}{l} \lambda_1 = 5 \\ \lambda_2 = 7 \end{array}$

PART II:

EIGENVALUES, EIGENVECTORS, EIGENSPACES
CASE II: REPEATED REAL EIGENVALUES
DEFECTIVE MATRICES

Case II: Repeated Real Eigenvalues (Motivation)

Consider the following diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A, B, C have the following eigenvalues:

$$\text{Matrix } A: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } B: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } C: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = 1 \end{array}$$

Moreover A, B, C have the following characteristic polynomials:

$$\begin{array}{l} \text{Matrix } A: \ p_A(\lambda) = \det(\lambda I - A) = (\lambda - 1)(\lambda - 2)(\lambda - 3) \\ \text{Matrix } B: \ p_B(\lambda) = \det(\lambda I - B) = (\lambda - 1)^2(\lambda - 3) \\ \text{Matrix } C: \ p_C(\lambda) = \det(\lambda I - C) = (\lambda - 1)^3 \end{array}$$

Notice repeated eigenvalues in B, C lead to repeated factors in $p_B(\lambda), p_C(\lambda)$.

Case II: Repeated Real Eigenvalues (Motivation)

Consider the following diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A, B, C have the following eigenvalues:

$$\text{Matrix } A: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } B: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } C: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = 1 \end{array}$$

Moreover, A has the following eigenspaces:

$$\text{Mtx } A: E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, E_{\lambda_3} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Finally, A has the following eigenvectors:

$$\text{Matrix } A: \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Case II: Repeated Real Eigenvalues (Motivation)

Consider the following diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A, B, C have the following eigenvalues:

$$\text{Matrix } A : \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } B : \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } C : \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = 1 \end{array}$$

Moreover, B has the following eigenspaces:

$$\text{Matrix } B : E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, E_{\lambda_3} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Finally, B has the following eigenvectors:

$$\text{Matrix } B : \mathbf{x}_{1,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_{1,2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Case II: Repeated Real Eigenvalues (Motivation)

Consider the following diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A, B, C have the following eigenvalues:

$$\text{Matrix } A: \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{matrix}, \quad \text{Matrix } B: \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = 3 \end{matrix}, \quad \text{Matrix } C: \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = 1 \end{matrix}$$

Moreover, C has the following eigenspaces:

$$\text{Matrix } C: E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Finally, C has the following eigenvectors:

$$\text{Matrix } C: \mathbf{x}_{1,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_{1,2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_{1,3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Multiplicities of an Eigenvalue (Definition)

When presented with repeated eigenvalue(s), multiplicities are useful:

Definition

(Multiplicities of an Eigenvalue)

Let matrix $A \in \mathbb{R}^{n \times n}$ have (repeated) real eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_p$
Moreover, let A have the following factored characteristic polynomial

$$p_A(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_p)^{m_p} \text{ (where } m_1, \ldots, m_p \in \mathbb{Z}_+\text{)}$$

The **algebraic multiplicity (AM)** of eigenvalue λ_k is m_k.

The **geometric multiplicity (GM)** of eigenvalue λ_k is $\dim(E_{\lambda_k})$.

i.e. $\text{AM}[\lambda_k] := m_k =$ # occurrences of $\lambda_k =$ power of factor $(\lambda - \lambda_k)$ in $p_A(\lambda)$.

i.e. $\text{GM}[\lambda_k] := \dim(E_{\lambda_k}) =$ # basis vectors of eigenspace E_{λ_k}.

NOTATION: $\mathbb{Z}_+ \equiv \{\text{positive integers}\} = \{1, 2, 3, 4, 5, \cdots\}$

Multiplicities of an Eigenvalue (Examples)

Consider the following diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A, B, C have the following eigenvalues:

$$\text{Matrix } A: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } B: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 3 \end{array}, \quad \text{Matrix } C: \quad \lambda_1 = 1$$

Moreover, A has the following characteristic polynomial:

$$\text{Matrix } A: \; p_A(\lambda) = \det(\lambda I - A) = (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

Then the eigenvalues of A have the following algebraic multiplicities:

$$\text{AM}[\lambda_1] = 1, \qquad \text{AM}[\lambda_2] = 1, \qquad \text{AM}[\lambda_3] = 1$$

Multiplicities of an Eigenvalue (Examples)

Consider the following diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A, B, C have the following eigenvalues:

$$\text{Matrix } A : \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } B : \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 3 \end{array}, \quad \text{Matrix } C : \quad \lambda_1 = 1$$

Moreover, B has the following characteristic polynomial:

$$\text{Matrix } B : \; p_B(\lambda) = \det(\lambda I - B) = (\lambda - 1)^2(\lambda - 3)$$

Then the eigenvalues of B have the following algebraic multiplicities:

$$\text{AM}[\lambda_1] = 2, \qquad \text{AM}[\lambda_2] = 1$$

Multiplicities of an Eigenvalue (Examples)

Consider the following diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A, B, C have the following eigenvalues:

$$\text{Matrix } A: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } B: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 3 \end{array}, \quad \text{Matrix } C: \quad \lambda_1 = 1$$

Moreover, C has the following characteristic polynomial:

$$\text{Matrix } C: \; p_C(\lambda) = \det(\lambda I - C) = (\lambda - 1)^3$$

Then the eigenvalue of C has the following algebraic multiplicity:

$$\text{AM}[\lambda_1] = 3$$

Multiplicities of an Eigenvalue (Examples)

Consider the following diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A, B, C have the following eigenvalues:

$$\text{Matrix } A: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } B: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 3 \end{array}, \quad \text{Matrix } C: \quad \lambda_1 = 1$$

Moreover, A has the following eigenspaces:

$$\text{Mtx } A: E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, E_{\lambda_3} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Then the eigenvalues of A have the following geometric multiplicities:

$$\text{GM}[\lambda_1] = 1, \quad \text{GM}[\lambda_2] = 1, \quad \text{GM}[\lambda_3] = 1$$

Multiplicities of an Eigenvalue (Examples)

Consider the following diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A, B, C have the following eigenvalues:

$$\text{Matrix } A: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } B: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 3 \end{array}, \quad \text{Matrix } C: \quad \lambda_1 = 1$$

Moreover, B has the following eigenspaces:

$$\text{Matrix } B : E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Then the eigenvalues of B have the following geometric multiplicities:

$$\text{GM}[\lambda_1] = 2, \qquad \text{GM}[\lambda_2] = 1$$

Multiplicities of an Eigenvalue (Examples)

Consider the following diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A, B, C have the following eigenvalues:

$$\text{Matrix } A: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } B: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 3 \end{array}, \quad \text{Matrix } C: \quad \lambda_1 = 1$$

Moreover, C has the following eigenspaces:

$$\text{Matrix } C: E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Then the eigenvalue of C has the following geometric multiplicity:

$$\text{GM}[\lambda_1] = 3$$

Multiplicities of an Eigenvalue (Examples)

Consider the following diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A, B, C have the following eigenvalues:

$$\text{Matrix } A: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } B: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 3 \end{array}, \quad \text{Matrix } C: \quad \lambda_1 = 1$$

To summarize the multiplicities for the eigenvalues of A:

$$\begin{array}{llll} \text{Matrix } A: & \text{AM}[\lambda_1] = 1, \quad \text{GM}[\lambda_1] = 1 & \Longrightarrow & \text{AM}[\lambda_1] = \text{GM}[\lambda_1] \\ \text{Matrix } A: & \text{AM}[\lambda_2] = 1, \quad \text{GM}[\lambda_2] = 1 & \Longrightarrow & \text{AM}[\lambda_2] = \text{GM}[\lambda_2] \\ \text{Matrix } A: & \text{AM}[\lambda_3] = 1, \quad \text{GM}[\lambda_3] = 1 & \Longrightarrow & \text{AM}[\lambda_3] = \text{GM}[\lambda_3] \end{array}$$

Is it always true that $\text{AM}[\lambda_k] = \text{GM}[\lambda_k]$???? Seems plausible...

Multiplicities of an Eigenvalue (Examples)

Consider the following diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A, B, C have the following eigenvalues:

$$\text{Matrix } A: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } B: \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 3 \end{array}, \quad \text{Matrix } C: \quad \lambda_1 = 1$$

To summarize the multiplicities for the eigenvalues of B:

$$\begin{array}{llll} \text{Matrix } B: & \text{AM}[\lambda_1] = 2, \quad \text{GM}[\lambda_1] = 2 & \Longrightarrow & \text{AM}[\lambda_1] = \text{GM}[\lambda_1] \\ \text{Matrix } B: & \text{AM}[\lambda_2] = 1, \quad \text{GM}[\lambda_2] = 1 & \Longrightarrow & \text{AM}[\lambda_2] = \text{GM}[\lambda_2] \end{array}$$

Is it always true that AM$[\lambda_k]$ = GM$[\lambda_k]$???? Seems plausible...

Multiplicities of an Eigenvalue (Examples)

Consider the following diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A, B, C have the following eigenvalues:

$$\text{Matrix } A: \quad \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}, \quad \text{Matrix } B: \quad \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 3 \end{array}, \quad \text{Matrix } C: \quad \lambda_1 = 1$$

To summarize the multiplicities for the eigenvalues of C:

$$\text{Matrix } C: \quad \text{AM}[\lambda_1] = 3, \quad \text{GM}[\lambda_1] = 3 \quad \Longrightarrow \quad \text{AM}[\lambda_1] = \text{GM}[\lambda_1]$$

Is it always true that $\text{AM}[\lambda_k] = \text{GM}[\lambda_k]$???? Seems plausible...

Multiplicities of an Eigenvalue (Shocking Examples!)

Consider the following upper triangular matrices:

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then D, E, F have the exact same eigenvalue & AM: $\lambda_1 = 1$
Moreover, D, E, F have the exact same characteristic polynomial:

$$p_D(\lambda) = p_E(\lambda) = p_F(\lambda) = (\lambda - \lambda_1)^3 = (\lambda - 1)^3 \implies \text{AM}[\lambda_1] = 3$$

Moreover, D has the following eigenspace:

$$\text{Matrix } D : E_{\lambda_1} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Then the eigenvalue of D has the following geometric multiplicity: GM$[\lambda_1] = 2$

Multiplicities of an Eigenvalue (Shocking Examples!)

Consider the following upper triangular matrices:

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then D, E, F have the exact same eigenvalue & AM: $\lambda_1 = 1$
Moreover, D, E, F have the exact same characteristic polynomial:

$$p_D(\lambda) = p_E(\lambda) = p_F(\lambda) = (\lambda - \lambda_1)^3 = (\lambda - 1)^3 \implies \text{AM}[\lambda_1] = 3$$

Moreover, E has the following eigenspace:

$$\text{Matrix } E : E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Then the eigenvalue of E has the following geometric multiplicity: $\text{GM}[\lambda_1] = 2$

Multiplicities of an Eigenvalue (Shocking Examples!)

Consider the following upper triangular matrices:

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then D, E, F have the exact same eigenvalue & AM: $\lambda_1 = 1$
Moreover, D, E, F have the exact same characteristic polynomial:

$$p_D(\lambda) = p_E(\lambda) = p_F(\lambda) = (\lambda - \lambda_1)^3 = (\lambda - 1)^3 \implies \text{AM}[\lambda_1] = 3$$

Moreover, F has the following eigenspace:

$$\text{Matrix } F : E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Then the eigenvalue of F has the following geometric multiplicity: $\text{GM}[\lambda_1] = 1$

Multiplicities of an Eigenvalue (Shocking Examples!)

Consider the following upper triangular matrices:

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then D, E, F have the exact same eigenvalue & AM: $\lambda_1 = 1$
Moreover, D, E, F have the exact same characteristic polynomial:

$$p_D(\lambda) = p_E(\lambda) = p_F(\lambda) = (\lambda - \lambda_1)^3 = (\lambda - 1)^3 \implies \text{AM}[\lambda_1] = 3$$

To summarize the multiplicities for the eigenvalues of D, E, F:

$$\begin{array}{lllll} \text{Matrix } D: & \text{AM}[\lambda_1] = 3, & \text{GM}[\lambda_1] = 2 & \implies & \text{AM}[\lambda_1] > \text{GM}[\lambda_1] \\ \text{Matrix } E: & \text{AM}[\lambda_1] = 3, & \text{GM}[\lambda_1] = 2 & \implies & \text{AM}[\lambda_1] > \text{GM}[\lambda_1] \\ \text{Matrix } F: & \text{AM}[\lambda_1] = 3, & \text{GM}[\lambda_1] = 1 & \implies & \text{AM}[\lambda_1] > \text{GM}[\lambda_1] \end{array}$$

Is it always true that $\text{AM}[\lambda_k] = \text{GM}[\lambda_k]$???? A resounding **NO!!!!!**

Defective Matrices (Definition)

So what matrices have eigenvalue(s) with differing AM & GM???

Definition

(Defective Matrix)

Let square matrix $A \in \mathbb{R}^{n \times n}$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$. Then:

A is a **defective matrix** if at least one eigenvalue λ_k satisfies AM$[\lambda_k] >$ GM$[\lambda_k]$

i.e. There's fewer linearly indep. eigenvectors for λ_k than # occurrences of λ_k.

The following matrices encountered earlier are defective:

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

CASE II: Repeated Real Eigenvalues (Procedure)

The procedure for CASE II is the same as for CASE I:

Proposition

(Find Eigenvalues, Eigenvectors, Eigenspaces – Repeated Real Eigenvalues)

GIVEN: Square Matrix $A \in \mathbb{R}^{n \times n}$ s.t. all eigenvalues are real, some repeated.

TASK: Find the Eigenvalues λ_k, Eigenvectors $\mathbf{x}_k$, Eigenspaces E_{λ_k} of A.

(1) Find Characteristic Polynomial $p_A(\lambda) = (-1)^n det(A - \lambda I)$

(2) Solve Characteristic Eqn $p_A(\lambda) = 0$ to find Eigenvalues $\lambda_1, \ldots, \lambda_p$ $(p < n)$

(3) Find the Eigenspace for each Eigenvalue λ_k: $E_{\lambda_k} = NullSp(A - \lambda_k I)$

(4) Find an Eigenvector for each λ_k.

If distinct λ_k: $\mathbf{x}_k = (\text{basis vector for } E_{\lambda_k})$

If repeated λ_k:

$\mathbf{x}_{k,1} = (1^{st} \text{ basis vector for } E_{\lambda_k})$, $\mathbf{x}_{k,2} = (2^{nd} \text{ basis vector for } E_{\lambda_k})$, $\ldots$

IMPORTANT: Repeated eigenvalues do not receive different indices!!

e.g. If A has eigenvalues $4, 2, 2, 2, -1, -1$, then: $\lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 4$

Invertibility & Eigenvalues

It turns out whether a matrix is invertible or not reveals whether zero is an eigenvalue or not:

Theorem

(Eigenvalues of Invertible & Non-Invertible Matrices)

Let square matrix $A \in \mathbb{R}^{n \times n}$. *Then:*

$$\begin{aligned} A \text{ is invertible} \quad &\iff \quad \text{All eigenvalues } \lambda_1, \lambda_2, \ldots, \lambda_p \text{ are non-zero} \\ A \text{ is not invertible} \quad &\iff \quad \text{At least one eigenvalue } \lambda_k = 0 \end{aligned}$$

PROOF: Omitted.

e.g. Consider the 2×2 matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

Then A has zero as an eigenvalue since A is not invertible. (identical columns)

Equivalent Conditions for **Singular** Square Matrices

Theorem

(Equivalent Conditions for Singular Square Matrices)

Let $A \in \mathbb{R}^{n \times n}$ be a ***square*** *matrix and $r < n$.*
Then the following are equivalent:

- *RREF(A) has r pivots*
- *rank$(A) = r$*
- *The rows of A are linearly dependent. Ditto for the columns of A.*
- *dim RowSp$(A) =$ dim ColSp$(A) = r$*
- *nullity$(A) = n - r$*
- *Linear system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has infinitely solutions $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h$*
- *Linear system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has infinitely many solutions only if $\vec{\mathbf{b}} \in$ ColSp(A)*
- *Linear system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has no solution only if $\vec{\mathbf{b}} \notin$ ColSp(A)*

- *A is not invertible (singular)*
- *det$(A) = 0$*
- ***A has at least one eigenvalue $\lambda_k = 0$***

PART III:

EIGENVALUES, EIGENVECTORS, EIGENSPACES

CASE III: SOME COMPLEX EIGENVALUES

CASE IV: ALL COMPLEX EIGENVALUES

Irreducible Quadratics

Definition

Let $a, b, c \in \mathbb{R}$. Then:

The **discriminant** of quadratic $ax^2 + bx + c$ is defined to be $b^2 - 4ac$.

Definition

Let $a, b, c \in \mathbb{R}$. Then:

Quadratic $ax^2 + bx + c$ is an **irreducible quadratic** $\iff b^2 - 4ac < 0$.

i.e., the linear factors of an irreducible quadratic are **complex** (not real):

(Recall that the **imaginary number** $i = \sqrt{-1}$.)

- $x^2 + 1$ is irreducible since $x^2 + 1 = (x - i)(x + i)$ $\quad \left[b^2 - 4ac = -4 < 0\right]$
- $x^2 - 1$ is reducible since $x^2 - 1 = (x - 1)(x + 1)$ $\quad \left[b^2 - 4ac = \quad 4 > 0\right]$
- $x^2 + 2x + 2$ is irreducible since $x^2 + 2x + 2 = [x + (1 - i)][x + (1 + i)]$ $\left[b^2 - 4ac = -4 < 0\right]$

Fundamental Theorem of Algebra (FTA)

Theorem

(Fundamental Theorem of Algebra)

*Every n^{th}-degree **polynomial with complex coefficients** can be factored into n **linear factors with complex coefficients**, some of which may be repeated.*

Corollary

*Every n^{th}-degree **polynomial with real coefficients** can be factored into **linears & irreducible quadratics with real coefficients**.*

What the corollary to the FTA means for finding eigenvalues is that the characteristic polynomial can always be factored into:

- Linear factors $(\lambda - \lambda_k)$
- Irreducible quadratics $(\lambda^2 + \alpha\lambda + \beta)$.

e.g. If a 4×4 matrix A has characteristic poly $p_A(\lambda) = (\lambda^2 + 1)(\lambda - 3)(\lambda + 4)$, then A has real eigenvalues $\lambda_1 = -4, \lambda_2 = 3$ and two complex eigenvalues since $\lambda^2 + 1$ is an irreducible quadratic.

Fundamental Theorem of Algebra (FTA)

Theorem

(Fundamental Theorem of Algebra)

*Every n^{th}-degree **polynomial with complex coefficients** can be factored into n **linear factors with complex coefficients**, some of which may be repeated.*

Corollary

*Every n^{th}-degree **polynomial with real coefficients** can be factored into **linears & irreducible quadratics with real coefficients**.*

REMARK: The FTA provides no procedure for factoring!

- $x^4 + 1 = \left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right)$
- $x^5 - 1 = (x - 1)\left(x^2 + \frac{1+\sqrt{5}}{2}x + 1\right)\left(x^2 + \frac{1-\sqrt{5}}{2}x + 1\right)$
- $x^5 + 1 = (x + 1)\left(x^2 - \frac{1+\sqrt{5}}{2}x + 1\right)\left(x^2 - \frac{1-\sqrt{5}}{2}x + 1\right)$
- $x^6 - 1 = (x - 1)(x + 1)\left(x^2 + x + 1\right)\left(x^2 - x + 1\right)$
- $x^6 + 1 = \left(x^2 + 1\right)\left(x^2 + \sqrt{3}x + 1\right)\left(x^2 - \sqrt{3}x + 1\right)$

CASE III: Some Complex Eigenvalues (Procedure)

For CASE III, just apply the procedure for CASE I/II, but ignore irreducible quadratics in the characteristic polynomial:

Proposition

(Find Eigenvalues, Eigenvectors, Eigenspaces – Some Complex Eigenvalues)

GIVEN: *Square Matrix* $A \in \mathbb{R}^{n \times n}$ *s.t. some eigenvalues are complex.*

TASK: *Find the* ***real*** *Eigenvalues* λ_k, *Eigenvectors* $\mathbf{x}_k$, *Eigenspaces* E_{λ_k} *of* A.

(1) Find Characteristic Polynomial $p_A(\lambda) = (-1)^n \mathit{det}(A - \lambda I)$

(2) Solve Characteristic Eqn $p_A(\lambda) = 0$, ***ignoring irreducible quadratics****, to find* ***real*** *Eigenvalues.*

(3) Find the Eigenspace for each real Eigenvalue λ_k: $E_{\lambda_k} = \mathit{NullSp}(A - \lambda_k I)$

(4) Find an Eigenvector for each λ_k.

If distinct λ_k: $\mathbf{x}_k = (\textit{basis vector for } E_{\lambda_k})$

If repeated λ_k:

$\mathbf{x}_{k,1} = (1^{st} \textit{ basis vector for } E_{\lambda_k})$, $\mathbf{x}_{k,2} = (2^{nd} \textit{ basis vector for } E_{\lambda_k})$, $\ldots$

CASE IV: All Complex Eigenvalues (Procedure)

The Good News: CASE IV will <u>never</u> be considered in this course!

The Bad News: CASE IV <u>will</u> show up in higher math courses (Diff Eqns II)

Here are some 2×2 matrices that have all complex eigenvalues:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \quad \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

The standard matrix for linear transformations representing certain <u>rotations</u> in an even-dimensional vector space like $\mathbb{R}^{2k}$ will have all complex eigenvalues.

Of course, since all matrices considered will have <u>real</u> entries, a complex eigenvalue will have complex eigenvector(s) by necessity.

Fin

Fin.