Sqaure Matrices: Diagonalization, Powers

Linear Algebra

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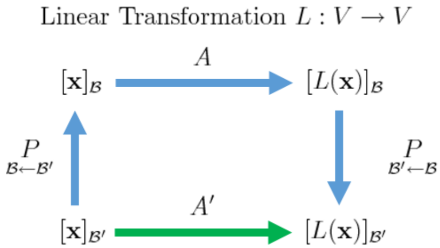
TTU

30 November 2015

Similar Matrices (Motivation)

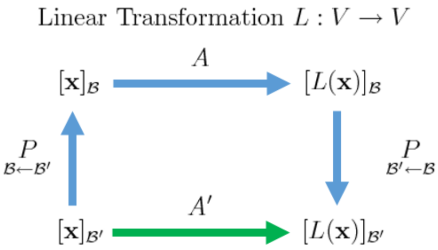


Consider the above **commutative diagram** of linear transformation L with standard matrices A, A' with respect to bases $\mathcal{B}, \mathcal{B}'$ of vector space V.

Traversing the above arrows translate into the following relations:

Left Blue Arrow:	$[\mathbf{x}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{B}'}{P}[\mathbf{x}]_{\mathcal{B}'}$
Top Blue Arrow:	$[L(\mathbf{x})]_{\mathcal{B}} = A[\mathbf{x}]_{\mathcal{B}}$
Right Blue Arrow:	$[L(\mathbf{x})]_{\mathcal{B}'} = \underset{\mathcal{B}'\leftarrow\mathcal{B}}{P}[L(\mathbf{x})]_{\mathcal{B}}$
Bottom Green Arrow:	$[L(\mathbf{x})]_{\mathcal{B}'} = A'[\mathbf{x}]_{\mathcal{B}'}$

Similar Matrices (Motivation)



Consider the above **commutative diagram** of linear transformation L with standard matrices A, A' with respect to bases $\mathcal{B}, \mathcal{B}'$ of vector space V.

One can start at $[\mathbf{x}]_{\mathcal{B}'}$ and end at $[L(\mathbf{x})]_{\mathcal{B}'}$ in two ways:

- Travel along the **blue** arrows: $[L(\mathbf{x})]_{\mathcal{B}'} = P^{-1}AP[\mathbf{x}]_{\mathcal{B}'}$
- Travel along the **green** arrow: $[L(\mathbf{x})]_{\mathcal{B}'} = A'[\mathbf{x}]_{\mathcal{B}'}$

where to reduce clutter simpler labels are used for the two transition matrices:

$$P \equiv \underset{\mathcal{B}\leftarrow\mathcal{B}'}{P} \implies P^{-1} \equiv \underset{\mathcal{B}'\leftarrow\mathcal{B}}{P}$$

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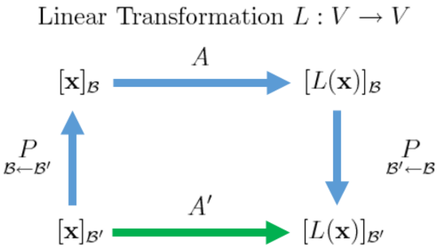

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The standard matrices A, A' are "similar" to each other in the sense that both apply the linear transformation L but using different coordinates/bases.

Equating the RHS's of the above formulas for $[L(\mathbf{x})]_{\mathcal{B}'}$ results in $A' = P^{-1}AP$

Similar Matrices (Definition)

So when do two $n \times n$ square matrices have the same eigenvalues??

Definition

(Similar Matrices)

Let square matrices $A, B \in \mathbb{R}^{n \times n}$.
Then A is **similar** to B if $\exists M \in \mathbb{R}^{n \times n}$ s.t. M is invertible and $A = M^{-1}BM$

$\exists \equiv$ "there exists"

Theorem

(Similar Matrices have the Same Eigenvalues)

Let square matrices $A, B \in \mathbb{R}^{n \times n}$. Then:

If A and B are similar, then A and B have the same eigenvalues.

Similar Matrices (Definition)

Theorem

(Similar Matrices have the Same Eigenvalues)

Let square matrices $A, B \in \mathbb{R}^{n\times n}$*. Then:*

If A *and* B *are similar, then* A *and* B *have the same eigenvalues.*

PROOF: Since matrices A & B are similar, $B = M^{-1}AM$ for some $M \in \mathbb{R}^{n\times n}$

$$\begin{aligned}
p_B(\lambda) &= |\lambda I - B| = |\lambda I - M^{-1}AM| && [I \text{ is } n \times n \text{ Identity Matrix}] \\
&= |(M^{-1}M)\lambda I - M^{-1}AM| && [\text{Clever Insertion of Identity: } I = M^{-1}M] \\
&= |M^{-1}\lambda IM - M^{-1}AM| && [MI = IM] \\
&= |M^{-1}(\lambda IM - AM)| && [\text{Left-Factor } M^{-1}] \\
&= |M^{-1}(\lambda I - A)M| && [\text{Right-Factor } M] \\
&= |M^{-1}||\lambda I - A||M| && [|ABC| = |AB||C| = |A||B||C|] \\
&= |M^{-1}||M||\lambda I - A| && [\text{Commutativity of Scalar Mult.: } ab = ba] \\
&= |M^{-1}M||\lambda I - A| && [\text{Determinant of Product: } |A||B| = |AB|] \\
&= |I||\lambda I - A| && [I = A^{-1}A = AA^{-1}] \\
&= |\lambda I - A| = p_A(\lambda) && [|I| = 1]
\end{aligned}$$

$\therefore\ p_B(\lambda) = p_A(\lambda) \implies A$ and B have the same eigenvalues QED

Diagonalizable Matrix (Definition)

Definition

(Diagonalizable Matrix – First Principles Defintion)

Let square matrix $A \in \mathbb{R}^{n \times n}$.
Then A is **diagonalizable** $\iff$ A is similar to an $n \times n$ diagonal matrix D.
i.e. A is diagonalizable if $\exists M \in \mathbb{R}^{n \times n}$ s.t. $M^{-1}AM = D$ where D is diagonal.

Theorem

(Equivalent Condition for Diagonalizability)

Let square matrix $A \in \mathbb{R}^{n \times n}$.
Then A is diagonalizable $\iff$ A has n linearly independent eigenvectors.

Theorem

(Sufficient Condition for Diagonalizability)

Let square matrix $A \in \mathbb{R}^{n \times n}$. Then:

If A has n distinct eigenvalues, then A is diagonalizable.

Diagonalization of a Square Matrix (Procedure)

Proposition

(Diagonalizing a Square Matrix)

GIVEN: Matrix $A \in \mathbb{R}^{n \times n}$ with all real eigenvalues, some possibly repeated.

TASK: Diagonalize Square Matrix A, if possible.

(1) Find the Eigenvalues of A: $\lambda_1, \ldots, \lambda_n$

(2) Find the Eigenspace E_{λ_k} for each unique Eigenvalue λ_k.
If AM$[\lambda_k] >$ GM$[\lambda_k]$, then A is not diagonalizable! STOP!!

(3) Find Eigenvector(s) for each unique Eigenvalue λ_k.

(4) Let matrix $X \in \mathbb{R}^{n \times n}$ s.t. its columns consist of the eigenvectors.

(5) Compute X^{-1}: $$[X \mid I] \xrightarrow{Gauss-Jordan} [I \mid X^{-1}]$$

(6) Let diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ s.t. the eigenvalues are on its main diagonal.
The order of eigenvectors in X determine the order of eigenvalues in Λ.

(7) Form the diagonalization of A: $$A = X \Lambda X^{-1}$$

NOTATION: Λ is the capital Greek letter 'lambda'.

Diagonalization of a Square Matrix (Orderings)

Consistency is key when ordering eigenvalues in Λ & eigenvectors in X:

Suppose 3×3 matrix A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. Then, A can be diagonalized as $A = X\Lambda X^{-1}$, where:

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

—— OR ——

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_2 & \mathbf{x}_1 & \mathbf{x}_3 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

—— OR ——

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_3 & \mathbf{x}_1 & \mathbf{x}_2 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

—— OR ——

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_3 & \mathbf{x}_2 & \mathbf{x}_1 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

Diagonalization of a Square Matrix (Orderings)

Consistency is key when ordering eigenvalues in Λ & eigenvectors in X:

Suppose 3×3 matrix A has eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{x}_{1,1}, \mathbf{x}_{1,2}, \mathbf{x}_2$. Then, A can be diagonalized as $A = X\Lambda X^{-1}$, where:

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— OR —

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_2 & \mathbf{x}_{1,1} & \mathbf{x}_{1,2} \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

Powers of Diagonal Matrices (Review)

Recall that powers of diagonal matrices are painless to find:

Proposition

(Powers of Diagonal Matrices)

*Let D be a $n \times n$ **diagonal matrix** and k be a **nonnegative integer**. Then:*

$$D^k = [d_{ij}^k \delta_{ij}]_{n\times n} = \begin{bmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \implies A^3 = \begin{bmatrix} 1^3 & 0 \\ 0 & 2^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \implies B^4 = \begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 0^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

The Value of Diagonalization (Powers Revisited)

Theorem

(Powers of a Square Matrix in Diagonalized Form)

*Let square matrix $A \in \mathbb{R}^{n \times n}$ such that A is **diagonalizable**:* $A = X\Lambda X^{-1}$
Moreover, let $k \in \overline{\mathbb{Z}}_+$ be a non-negative integer. Then: $A^k = X\Lambda^k X^{-1}$

PROOF: (I is the $n \times n$ identity matrix.)

$$
\begin{array}{ccccl}
A^0 & = & I & = & XX^{-1} = XIX^{-1} = X\Lambda^0 X^{-1} \\
A^1 & = & A & = & X\Lambda X^{-1} = X\Lambda^1 X^{-1} \\
A^2 & = & AA & = & (X\Lambda X^{-1})(X\Lambda X^{-1}) = X\Lambda(X^{-1}X)\Lambda X^{-1} \\
 & & & = & X\Lambda I\Lambda X^{-1} = X\Lambda\Lambda X^{-1} = X\Lambda^2 X^{-1} \\
A^3 & = & A^2 A & = & (X\Lambda^2 X^{-1})(X\Lambda X^{-1}) = X\Lambda^2(X^{-1}X)\Lambda X^{-1} \\
 & & & = & X\Lambda^2 I\Lambda X^{-1} = X\Lambda^2\Lambda X^{-1} = X\Lambda^3 X^{-1} \\
\vdots & & \vdots & & \qquad\qquad\vdots \\
A^{k-1} & = & A^{k-2}A & = & \cdots = X\Lambda^{k-1}X^{-1} \\
A^k & = & A^{k-1}A & = & (X\Lambda^{k-1}X^{-1})(X\Lambda X^{-1}) = X\Lambda^{k-1}(X^{-1}X)\Lambda X^{-1} \\
 & & & = & X\Lambda^{k-1}I\Lambda X^{-1} = X\Lambda^{k-1}\Lambda X^{-1} = X\Lambda^k X^{-1}
\end{array}
$$

QED

The Value of Diagonalization (Powers Revisited)

The power of an $n \times n$ square matrix in diagonalized form requires far less work, especially if the matrix is **large** $(n \geq 3)$ and **dense**:

Power by Brute Force: $A^{10} = AAAAAAAAAA$
Power of Diagonalized Form: $A^{10} = X\Lambda^{10}X^{-1}$

	POWER BY BRUTE FORCE	**POWER OF DIAGONALIZED FORM**
k	DENSE MATRIX MULTIPLICATIONS	DENSE MATRIX MULTIPLICATIONS
2	1	1
3	2	1
4	3	1
5	4	1
⋮	⋮	⋮
10	9	1
⋮	⋮	⋮
1000	**999**	1

Powers of diagonalized forms require far less work for computers.

Eigenvalues & Eigenvectors of the Inverse of a Matrix

If the eigenpairs $(\lambda, \mathbf{x})$ of invertible matrix A are known,
then the eigenpairs of its inverse A^{-1} can be quickly found:

Theorem

(Eigenvalues & Eigenvectors of the Inverse of a Matrix)

Let square matrix $A \in \mathbb{R}^{n \times n}$ be invertible. Then:

$$(\lambda, \mathbf{x}) \text{ is an eigenpair of } A \iff \left(\frac{1}{\lambda}, \mathbf{x}\right) \text{ is an eigenpair of } A^{-1}$$

PROOF: Let matrix A be invertible. Then, $\lambda \neq 0$.

$(\lambda, \mathbf{x})$ is an eigenpair of $A \iff A\mathbf{x} = \lambda\mathbf{x} \iff \mathbf{x} = \lambda A^{-1}\mathbf{x}$

$\iff \frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x} \iff A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x} \iff \left(\frac{1}{\lambda}, \mathbf{x}\right)$ is an eigenpair of A^{-1}

QED

Fin.