Symmetric Matrices, Orthogonal Matrices

Linear Algebra

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TTU

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PART I:

SYMMETRIC MATRICES

ORTHOGONAL MATRICES

Symmetric Matrices (Definition)

The notion of a symmetric matrix is fundamental for later concepts & courses:

Definition

(Symmetric Matrix)

A square matrix $A \in \mathbb{R}^{n \times n}$ is **symmetric** if A is equal to its transpose: $A^T = A$

Corollary

(Diagonal Matrices are Symmetric)

A diagonal matrix $D \in \mathbb{R}^{n \times n}$ is symmetric.

Symmetric Matrices: $\begin{bmatrix} 1 & \mathbf{2} \\ \mathbf{2} & 3 \end{bmatrix}, \begin{bmatrix} -4 & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & 4 & \mathbf{5} \\ \mathbf{3} & \mathbf{5} & 6 \end{bmatrix}$

Not Symmetric: $\begin{bmatrix} -4 & \mathbf{0} \\ \mathbf{1} & -1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{2} & \mathbf{3} \\ \mathbf{9} & 4 & \mathbf{5} \\ \mathbf{8} & \mathbf{0} & 6 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{2} & \mathbf{3} \\ \mathbf{9} & 4 & \mathbf{5} \\ \mathbf{3} & \mathbf{5} & 6 \end{bmatrix}$

Symmetric Matrices (Properties)

Symmetric matrices have some very nice properties:

Theorem

(The Real Spectrum Theorem)

Let symmetric matrix $S \in \mathbb{R}^{n \times n}$. *Then the following all hold:*

- S *is diagonalizable*
- *All eigenvalues of* S *are real*
- *If some eigenvalue* λ_k *repeats, its multiplicities match:* $AM[\lambda_k] = GM[\lambda_k]$
  i.e. If λ_k *occurs* j *times, then* λ_k *has* j *linearly independent eigenvectors:*

$$\mathbf{x}_{k,1},\ \mathbf{x}_{k,2},\ \ldots,\ \mathbf{x}_{k,j-1},\ \mathbf{x}_{k,j}$$

PROOF: It's complicated...

Orthogonal Matrices (Definition)

Question: When is the inverse of a square matrix is simply its transpose??
Answer: When the square matrix is orthogonal:

Definition

(Orthogonal Matrix)

A square matrix $Q \in \mathbb{R}^{n\times n}$ is **orthogonal** if Q is invertible and $Q^{-1} = Q^T$

Corollary

(Determining if a Matrix is Orthogonal)

A square matrix $Q \in \mathbb{R}^{n\times n}$ is ***orthogonal*** $\iff$ $Q^T Q = I$

Theorem

(An Orthogonal Matrix has Orthonormal Columns)

A square matrix Q is ***orthogonal*** $\iff$ *its columns form an orthonormal set.*

PROOF: See the textbook if interested.

Orthogonal Matrices (Properties)

The following theorem is the cornerstone to many stable numerical algorithms involving orthogonal matrices:

Theorem

(Orthogonal Preservation Theorem)

Consider the Euclidean inner product space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ *where* $\mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathbb{R}^n$ *and*

Inner product	$\langle \mathbf{v}, \mathbf{w}\rangle$	$:=$	$\mathbf{v}^T\mathbf{w}$
Induced norm	$\|\|\mathbf{x}\|\|$	$:=$	$\sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$
Induced metric	$d(\mathbf{v}, \mathbf{w})$	$:=$	$\|\|\mathbf{v} - \mathbf{w}\|\|$

Then orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ *preserves inner products, norms & metrics:*

$$(i)\ \langle Q\mathbf{v}, Q\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle, \qquad (ii)\ ||Q\mathbf{x}|| = ||\mathbf{x}||, \qquad (iii)\ d(Q\mathbf{v}, Q\mathbf{w}) = d(\mathbf{v}, \mathbf{w})$$

Orthogonal Matrices (Properties)

Theorem

(Orthogonal Preservation Theorem)

Consider the Euclidean inner product space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ *where* $\mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathbb{R}^n$ *and*

Inner product	$\langle \mathbf{v}, \mathbf{w} \rangle$	$:=$	$\mathbf{v}^T\mathbf{w}$
Induced norm	$\lVert\mathbf{x}\rVert$	$:=$	$\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$
Induced metric	$d(\mathbf{v}, \mathbf{w})$	$:=$	$\lVert\mathbf{v} - \mathbf{w}\rVert$

Then orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ *preserves inner products, norms & metrics:*

$(i)\ \langle Q\mathbf{v}, Q\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle,$ $\quad (ii)\ \lVert Q\mathbf{x}\rVert = \lVert\mathbf{x}\rVert,$ $\quad (iii)\ d(Q\mathbf{v}, Q\mathbf{w}) = d(\mathbf{v}, \mathbf{w})$

PROOF:

$$(i)\ \ \langle Q\mathbf{v}, Q\mathbf{w} \rangle := (Q\mathbf{v})^T(Q\mathbf{w}) \overset{T4}{=} \mathbf{v}^T(Q^TQ)\mathbf{w} = \mathbf{v}^T(Q^{-1}Q)\mathbf{w} = \mathbf{v}^TI\mathbf{w} = \mathbf{v}^T\mathbf{w} := \langle \mathbf{v}, \mathbf{w} \rangle$$

$$(ii)\ \ \lVert Q\mathbf{x}\rVert^2 = \langle Q\mathbf{x}, Q\mathbf{x} \rangle \overset{(i)}{=} \langle \mathbf{x}, \mathbf{x} \rangle = \lVert\mathbf{x}\rVert^2 \implies \lVert Q\mathbf{x}\rVert = \lVert\mathbf{x}\rVert$$

$$(iii)\ d(Q\mathbf{v}, Q\mathbf{w}) := \lVert Q\mathbf{v} - Q\mathbf{w}\rVert \overset{M3}{=} \lVert Q(\mathbf{v} - \mathbf{w})\rVert \overset{(ii)}{=} \lVert\mathbf{v} - \mathbf{w}\rVert$$

QED

Eigenvectors of a Symmetric Matrix

Eigenvectors of distinct eigenvalues of a symmetric matrix have a benefit:

Theorem

(Eigenvectors of a Symmetric Matrix)

Let symmetric matrix $S \in \mathbb{R}^{n \times n}$ *have eigenpairs* $(\lambda_1, \mathbf{x}_1), (\lambda_2, \mathbf{x}_2)$. *Then:*

If eigenvalues λ_1, λ_2 *are distinct, then eigenvectors* $\mathbf{x}_1, \mathbf{x}_2$ *are orthogonal.*

i.e. If $\lambda_1 \neq \lambda_2$, *then* $\mathbf{x}_1 \perp \mathbf{x}_2$.

Eigenvectors of a Symmetric Matrix

Theorem

(Eigenvectors of a Symmetric Matrix)

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If eigenvalues λ_1, λ_2 *are distinct, then eigenvectors* $\mathbf{x}_1, \mathbf{x}_2$ *are orthogonal.*

i.e. If $\lambda_1 \neq \lambda_2$, *then* $\mathbf{x}_1 \perp \mathbf{x}_2$.

PROOF: Let $\begin{matrix}(\lambda_1, \mathbf{x}_1)\\(\lambda_2, \mathbf{x}_2)\end{matrix}$ be eigenpairs of S s.t. $\lambda_1 \neq \lambda_2$. Then $\begin{matrix}S\mathbf{x}_1 = \lambda_1\mathbf{x}_1\\ S\mathbf{x}_2 = \lambda_2\mathbf{x}_2\end{matrix}$

Moreover: (Since S is symmetric, $S = S^T$)

$$\lambda_1(\mathbf{x}_1^T\mathbf{x}_2) = (\lambda_1\mathbf{x}_1)^T\mathbf{x}_2 \overset{EIG}{=} (S\mathbf{x}_1)^T\mathbf{x}_2 \overset{T4}{=} \mathbf{x}_1^T S^T\mathbf{x}_2 \overset{SYM}{=} \mathbf{x}_1^T(S\mathbf{x}_2) \overset{EIG}{=} \mathbf{x}_1^T(\lambda_2\mathbf{x}_2) = \lambda_2(\mathbf{x}_1^T\mathbf{x}_2)$$

Hence:

$$\lambda_1(\mathbf{x}_1^T\mathbf{x}_2) = \lambda_2(\mathbf{x}_1^T\mathbf{x}_2) \implies \lambda_1(\mathbf{x}_1^T\mathbf{x}_2) - \lambda_2(\mathbf{x}_1^T\mathbf{x}_2) = 0 \implies (\lambda_1 - \lambda_2)(\mathbf{x}_1^T\mathbf{x}_2) = 0$$

Since $\lambda_1 \neq \lambda_2 \iff \lambda_1 - \lambda_2 \neq 0$, the equation $(\lambda_1 - \lambda_2)(\mathbf{x}_1^T\mathbf{x}_2) = 0$ implies that $\mathbf{x}_1^T\mathbf{x}_2 = 0 \implies \mathbf{x}_1 \perp \mathbf{x}_2 \implies \mathbf{x}_1, \mathbf{x}_2$ are orthogonal QED

PART II:

ORTHOGONAL DIAGONALIZATION OF A SYMMETRIC MATRIX

Orthogonally Diagonalizable Matrices

Definition

(Orthogonally Diagonalizable Matrix)

Let square matrix $A \in \mathbb{R}^{n \times n}$.
Then A is **orthogonally diagonalizable** if $\exists Q \in \mathbb{R}^{n \times n}$ s.t. Q is orthogonal and

$$Q^T A Q = D \text{ where } D \text{ is diagonal.}$$

Theorem

(Symmetric Matrices are Orthogonally Diagonalizable)

Square matrix A is orthgonally diagonalizable $\iff$ A is symmetric.

PROOF: $(\implies)$: Let A be orthognally diagonalizable.
Then $Q^T A Q = D$ for some orthogonal matrix Q and diagonal matrix D.
Now, $Q^T A Q = D \implies Q Q^T A Q Q^T = Q D Q^T \implies IAI = QDQ^T \implies A = QDQ^T$
Now, $A^T = (QDQ^T)^T \overset{T4}{=} Q^{TT} D^T Q^T = Q D^T Q^T \overset{SYM}{=} QDQ^T = A$
$\therefore\ A^T = A \implies A$ is symmetric
$(\Leftarrow)$: It's subtle – see the textbook if interested. QED

Orthogonal Diagonalization of a Symmetric Matrix

Proposition

(Orthogonally Diagonalizing a Symmetric Matrix)

GIVEN: Symmetric Matrix $S \in \mathbb{R}^{n \times n}$ with some possibly repeated eigenvalues.

TASK: Orthogonally Diagonalize Symmetric Matrix S.

(1) Find the Eigenvalues of S: $\lambda_1, \ldots, \lambda_n$

(2) Find the Eigenspace E_{λ_k} for each unique Eigenvalue λ_k.

(3) Find Unit Eigenvector(s) $\widehat{\mathbf{q}}_k$ for each unique Eigenvalue λ_k: $\widehat{\mathbf{q}}_k = \frac{\mathbf{q}_k}{||\mathbf{q}_k||}$
If AM$[\lambda_k] \geq 2$, then apply Gram-Schmidt on the eigenvectors for λ_k.

(4) Let matrix $Q \in \mathbb{R}^{n \times n}$ s.t. its columns consist of the unit eigenvectors.

(5) Let diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ s.t. the eigenvalues are on its main diagonal.
The order of eigenvectors in Q determine the order of eigenvalues in Λ.

(6) Form the diagonalization of S: $S = Q \Lambda Q^T$

NOTATION: Λ is the capital Greek letter 'lambda'.

Ortho. Diagonalization of Symmetric Matrix (Ordering)

Consistency is key when ordering eigenvalues in Λ & eigenvectors in Q:

Let 3×3 sym. matrix S have eigenvalues $\lambda_1, \lambda_2, \lambda_3$ & eigenvectors $\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2, \widehat{\mathbf{q}}_3$. Then, S can be orthogonally diagonalized as $S = Q\Lambda Q^T$, where:

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_1 & \widehat{\mathbf{q}}_2 & \widehat{\mathbf{q}}_3 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

— OR —

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_2 & \widehat{\mathbf{q}}_1 & \widehat{\mathbf{q}}_3 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

— OR —

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_3 & \widehat{\mathbf{q}}_1 & \widehat{\mathbf{q}}_2 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

— OR —

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_3 & \widehat{\mathbf{q}}_2 & \widehat{\mathbf{q}}_1 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

Ortho. Diagonalization of Symmetric Matrix (Ordering)

Consistency is key when ordering eigenvalues in Λ & eigenvectors in Q:

Let 3×3 sym. matrix S have eigenvalues λ_1, λ_2 & eigenvectors $\widehat{\mathbf{q}}_{1,1}, \widehat{\mathbf{q}}_{1,2}, \widehat{\mathbf{q}}_2$. Then, S can be orthogonally diagonalized as $S = Q\Lambda Q^T$, where:

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_{1,1} & \widehat{\mathbf{q}}_{1,2} & \widehat{\mathbf{q}}_2 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

— OR —

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_{1,2} & \widehat{\mathbf{q}}_{1,1} & \widehat{\mathbf{q}}_2 \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

— OR —

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_{1,1} & \widehat{\mathbf{q}}_2 & \widehat{\mathbf{q}}_{1,2} \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

— OR —

$$Q = \begin{bmatrix} | & | & | \\ \widehat{\mathbf{q}}_2 & \widehat{\mathbf{q}}_{1,1} & \widehat{\mathbf{q}}_{1,2} \\ | & | & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

Fin

Fin.