Householder Reflectors: $A = QR$

Linear Algebra

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TTU

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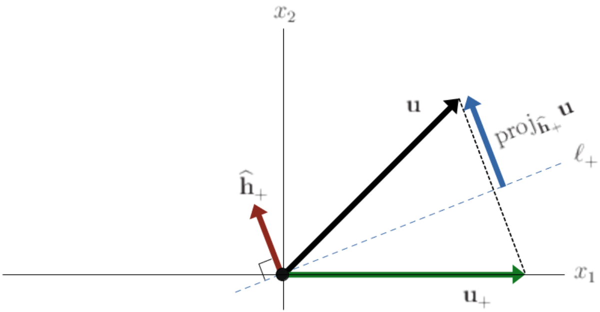
PART I:

Householder Reflectors in $\mathbb{R}^2$
Householder Reflectors in $\mathbb{R}^n$

Properties of Householder Reflector Matrices

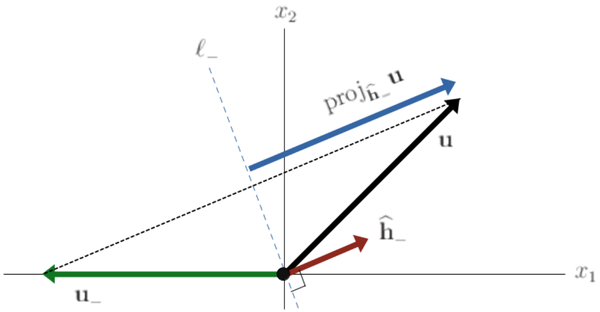
Householder Reflection onto Same x_1-Semiaxis in $\mathbb{R}^2$ (Quadrant I)



How to reflect a vector $\mathbf{u}$ in Quadrant I onto the same x_1-semiaxis, denoted as $\mathbf{u}_+$:

1. Observe that $\mathbf{u}_+ = +1 \cdot ||\mathbf{u}||_2 \cdot \hat{\mathbf{e}}_1$ since reflections do not alter magnitude.
2. Form bisecting ray, ℓ_+, from origin outward into the same x_1-halfplane as $\mathbf{u}$.
3. Form & normalize vector $\mathbf{h}_+$ orthogonal to ℓ_+ and pointing toward same x_2-halfplane as $\mathbf{u}$.
4. Project vector $\mathbf{u}$ onto unit vector $\hat{\mathbf{h}}_+$.
5. Subtract twice this projection from vector $\mathbf{u}$, resulting in $\mathbf{u}_+ = \mathbf{u} - 2 \cdot \text{proj}_{\hat{\mathbf{h}}_+} \mathbf{u}$.

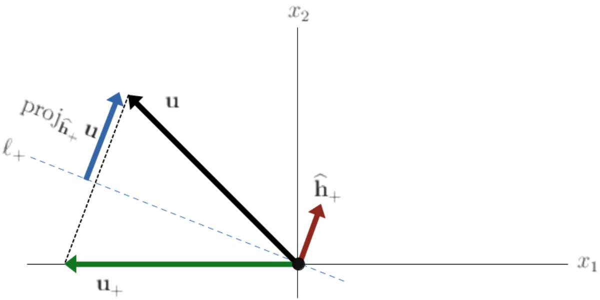
Householder Reflection onto Other x_1-Semiaxis in $\mathbb{R}^2$ (Quadrant I)



How to reflect a vector $\mathbf{u}$ in Quadrant I onto the other x_1-semiaxis, denoted as $\mathbf{u}_-$:

1. Observe that $\mathbf{u}_- = -1 \cdot ||\mathbf{u}||_2 \cdot \hat{\mathbf{e}}_1$ since reflections do not alter magnitude.
2. Form bisecting ray, ℓ_-, from origin outward into the other x_1-halfplane as $\mathbf{u}$.
3. Form & normalize vector $\mathbf{h}_-$ orthogonal to ℓ_- and pointing toward same x_2-halfplane as $\mathbf{u}$.
4. Project vector $\mathbf{u}$ onto unit vector $\hat{\mathbf{h}}_-$.
5. Subtract twice this projection from vector $\mathbf{u}$, resulting in $\mathbf{u}_- = \mathbf{u} - 2 \cdot \text{proj}_{\hat{\mathbf{h}}_-}\mathbf{u}$.

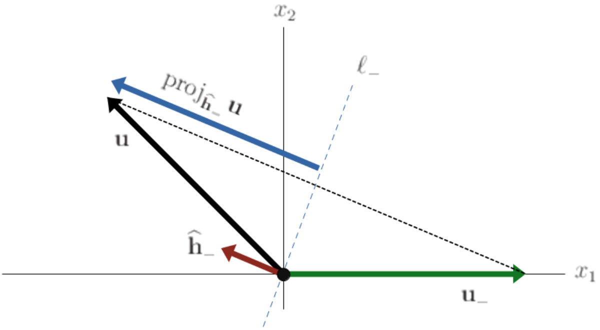
Householder Reflection onto Same x_1-Semiaxis in $\mathbb{R}^2$ (Quadrant II)



How to reflect a vector $\mathbf{u}$ in Quadrant II onto the same x_1-semiaxis, denoted as $\mathbf{u}_+$:

1. Observe that $\mathbf{u}_+ = -1 \cdot ||\mathbf{u}||_2 \cdot \hat{\mathbf{e}}_1$ since reflections do not alter magnitude.
2. Form bisecting ray, ℓ_+, from origin outward into the same x_1-halfplane as $\mathbf{u}$.
3. Form & normalize vector $\mathbf{h}_+$ orthogonal to ℓ_+ and pointing toward same x_2-halfplane as $\mathbf{u}$.
4. Project vector $\mathbf{u}$ onto unit vector $\hat{\mathbf{h}}_+$.
5. Subtract twice this projection from vector $\mathbf{u}$, resulting in $\mathbf{u}_+ = \mathbf{u} - 2 \cdot \text{proj}_{\hat{\mathbf{h}}_+}\mathbf{u}$.

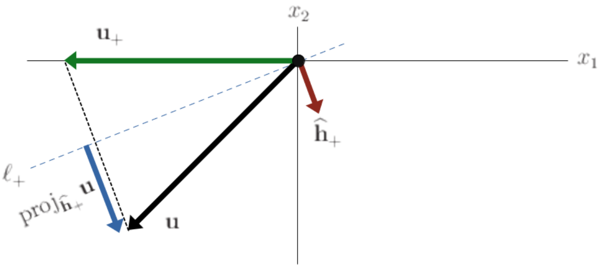
Householder Reflection onto Other x_1-Semiaxis in $\mathbb{R}^2$ (Quadrant II)



How to reflect a vector $\mathbf{u}$ in Quadrant II onto the other x_1-semiaxis, denoted as $\mathbf{u}_-$:

1. Observe that $\mathbf{u}_- = +1 \cdot ||\mathbf{u}||_2 \cdot \hat{\mathbf{e}}_1$ since reflections do not alter magnitude.
2. Form bisecting ray, ℓ_-, from origin outward into the other x_1-halfplane as $\mathbf{u}$.
3. Form & normalize vector $\mathbf{h}_-$ orthogonal to ℓ_- and pointing toward same x_2-halfplane as $\mathbf{u}$.
4. Project vector $\mathbf{u}$ onto unit vector $\hat{\mathbf{h}}_-$.
5. Subtract twice this projection from vector $\mathbf{u}$, resulting in $\mathbf{u}_- = \mathbf{u} - 2 \cdot \text{proj}_{\hat{\mathbf{h}}_-}\,\mathbf{u}$.

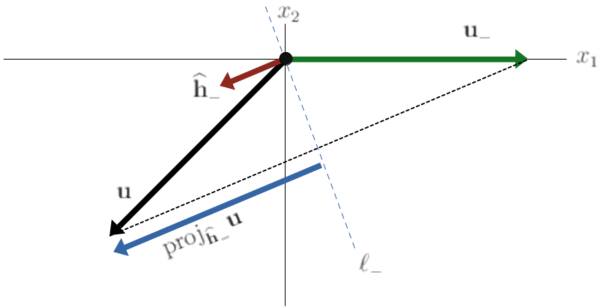
Householder Reflection onto Same x_1-Semiaxis in $\mathbb{R}^2$ (Quadrant III)



How to reflect a vector $\mathbf{u}$ in Quadrant III onto the same x_1-semiaxis, denoted as $\mathbf{u}_+$:

1. Observe that $\mathbf{u}_+ = -1 \cdot ||\mathbf{u}||_2 \cdot \hat{\mathbf{e}}_1$ since reflections do not alter magnitude.
2. Form bisecting ray, ℓ_+, from origin outward into the same x_1-halfplane as $\mathbf{u}$.
3. Form & normalize vector $\mathbf{h}_+$ orthogonal to ℓ_+ and pointing toward same x_2-halfplane as $\mathbf{u}$.
4. Project vector $\mathbf{u}$ onto unit vector $\hat{\mathbf{h}}_+$.
5. Subtract twice this projection from vector $\mathbf{u}$, resulting in $\mathbf{u}_+ = \mathbf{u} - 2 \cdot \text{proj}_{\hat{\mathbf{h}}_+} \mathbf{u}$.

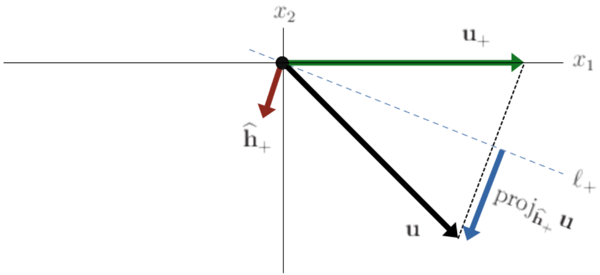
Householder Reflection onto Other x_1-Semiaxis in $\mathbb{R}^2$ (Quadrant III)



How to reflect a vector $\mathbf{u}$ in Quadrant III onto the other x_1-semiaxis, denoted as $\mathbf{u}_-$:

1. Observe that $\mathbf{u}_- = +1 \cdot ||\mathbf{u}||_2 \cdot \hat{\mathbf{e}}_1$ since reflections do not alter magnitude.
2. Form bisecting ray, ℓ_-, from origin outward into the other x_1-halfplane as $\mathbf{u}$.
3. Form & normalize vector $\mathbf{h}_-$ orthogonal to ℓ_- and pointing toward same x_2-halfplane as $\mathbf{u}$.
4. Project vector $\mathbf{u}$ onto unit vector $\hat{\mathbf{h}}_-$.
5. Subtract twice this projection from vector $\mathbf{u}$, resulting in $\mathbf{u}_- = \mathbf{u} - 2 \cdot \mathrm{proj}_{\hat{\mathbf{h}}_-}\mathbf{u}$.

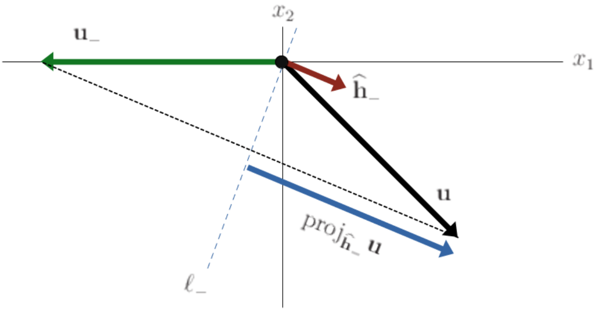
Householder Reflection onto Same x_1-Semiaxis in $\mathbb{R}^2$ (Quadrant IV)



How to reflect a vector $\mathbf{u}$ in Quadrant IV onto the same x_1-semiaxis, denoted as $\mathbf{u}_+$:

1. Observe that $\mathbf{u}_+ = +1 \cdot ||\mathbf{u}||_2 \cdot \hat{\mathbf{e}}_1$ since reflections do not alter magnitude.
2. Form bisecting ray, ℓ_+, from origin outward into the same x_1-halfplane as $\mathbf{u}$.
3. Form & normalize vector $\mathbf{h}_+$ orthogonal to ℓ_+ and pointing toward same x_2-halfplane as $\mathbf{u}$.
4. Project vector $\mathbf{u}$ onto unit vector $\hat{\mathbf{h}}_+$.
5. Subtract twice this projection from vector $\mathbf{u}$, resulting in $\mathbf{u}_+ = \mathbf{u} - 2 \cdot \text{proj}_{\hat{\mathbf{h}}_+} \mathbf{u}$.

Householder Reflection onto Other x_1-Semiaxis in $\mathbb{R}^2$ (Quadrant IV)



How to reflect a vector $\mathbf{u}$ in Quadrant IV onto the other x_1-semiaxis, denoted as $\mathbf{u}_-$:

1. Observe that $\mathbf{u}_- = -1 \cdot ||\mathbf{u}||_2 \cdot \hat{\mathbf{e}}_1$ since reflections do not alter magnitude.
2. Form bisecting ray, ℓ_-, from origin outward into the other x_1-halfplane as $\mathbf{u}$.
3. Form & normalize vector $\mathbf{h}_-$ orthogonal to ℓ_- and pointing toward same x_2-halfplane as $\mathbf{u}$.
4. Project vector $\mathbf{u}$ onto unit vector $\hat{\mathbf{h}}_-$.
5. Subtract twice this projection from vector $\mathbf{u}$, resulting in $\mathbf{u}_- = \mathbf{u} - 2 \cdot \text{proj}_{\hat{\mathbf{h}}_-} \mathbf{u}$.

Signum Function

Definition

(Signum Function)

$$\text{sign}(x) := \begin{cases} +1 & \text{, if } x > 0 \\ 0 & \text{, if } x = 0 \\ -1 & \text{, if } x < 0 \end{cases}$$

Signum Function

The signum function can be modified so that zero is forced to be either $+1$ or -1:

Definition

(Upper-Signum Function)

$$\overline{\text{sign}}(x) := \begin{cases} +1 & \text{, if } x \geq 0 \\ -1 & \text{, if } x < 0 \end{cases}$$

Definition

(Lower-Signum Function)

$$\underline{\text{sign}}(x) := \begin{cases} +1 & \text{, if } x > 0 \\ -1 & \text{, if } x \leq 0 \end{cases}$$

For Householder Reflectors, it's convenient to use the upper-signum function.

Same-Semiaxis Householder Reflectors in $\mathbb{R}^n$

Proposition

(Same-Semiaxis Householder Reflectors)

Let vector $\mathbf{u} \in \mathbb{R}^n$. *Then:*

$$\begin{aligned}
(1) \quad \mathbf{h}_+ &= \mathbf{u} - \overline{\mathit{sign}}(u_1) \cdot ||\mathbf{u}||_2 \cdot \hat{\mathbf{e}}_1 \\
(2) \quad \hat{\mathbf{h}}_+ &= \mathbf{h}_+ / ||\mathbf{h}_+||_2 \\
(3) \quad \ell_+ &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{h}_+ \cdot \mathbf{x} = 0\}
\end{aligned}$$

$$\begin{aligned}
(4) \quad \bar{P}_+ &= \hat{\mathbf{h}}_+ \hat{\mathbf{h}}_+^T \\
(5) \quad H_+ &= I - 2\bar{P}_+
\end{aligned}$$

$$(6) \quad \mathbf{u}_+ = H_+ \mathbf{u} = \mathbf{u} - 2 \cdot \mathit{proj}_{\hat{\mathbf{h}}_+} \mathbf{u}$$

SANITY CHECKS: H_+ *is symmetric and orthogonal.*

Proposition

(Other-Semiaxis Householder Reflectors)

Let vector $\mathbf{u} \in \mathbb{R}^n$. *Then:*

$$
\begin{aligned}
&(1) \quad \mathbf{h}_- = \mathbf{u} + \overline{\mathit{sign}}(u_1) \cdot ||\mathbf{u}||_2 \cdot \hat{\mathbf{e}}_1 \\
&(2) \quad \hat{\mathbf{h}}_- = \mathbf{h}_- / ||\mathbf{h}_-||_2 \\
&(3) \quad \ell_- = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{h}_- \cdot \mathbf{x} = 0\} \\
\\
&(4) \quad \bar{P}_- = \hat{\mathbf{h}}_- \hat{\mathbf{h}}_-^T \\
&(5) \quad H_- = I - 2\bar{P}_- \\
\\
&(6) \quad \mathbf{u}_- = H_- \mathbf{u} = \mathbf{u} - 2 \cdot \mathit{proj}_{\hat{\mathbf{h}}_-} \mathbf{u}
\end{aligned}
$$

SANITY CHECKS: H_- *is symmetric and orthogonal.*

Properties of Householder Reflector Matrices

Proposition

(Properties of Householder Reflector Matrices)

Let $H := \hat{\mathbf{h}}\hat{\mathbf{h}}^T \in \mathbb{R}^{n\times n}$ *be a Householder Reflector matrix. Then:*

(1) H *is symmetric:* $H^T = H$

(2) H *is orthogonal:* $H^{-1} = H^T$

(3) H *is involutory:* $H^2 = I$

(4) $H\hat{\mathbf{h}} = -\hat{\mathbf{h}}$

(5) $H\mathbf{h}_\perp = \mathbf{h}_\perp \ \forall \mathbf{h}_\perp \in \{\hat{\mathbf{h}}\}^\perp$

(6) *The eigenvalues of* H *are:* $-1; \ \underbrace{1, 1, \cdots, 1}_{n-1}$

(7) *det*$(H) = -1$

PROOF: Left as an exercise for the reader.

PART II:

Full QR Factorization via Householder Reflectors

Full QR Factorization via Householder Reflectors

Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then:

$$\mathbf{a}_1 := (a_{11}, a_{21})^T \implies \mathbf{h}_1 = \mathbf{a}_1 \pm \overline{\text{sign}}(a_{11}) \cdot ||\mathbf{a}_1||_2 \cdot \hat{\mathbf{e}}_1 \implies \hat{\mathbf{h}}_1 = \mathbf{h}_1 / ||\mathbf{h}_1||_2$$

$$\implies H_1' = I - 2\hat{\mathbf{h}}_1\hat{\mathbf{h}}_1^T \implies H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = H_1'$$

$$H_1 A = \begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}' \end{bmatrix} = R$$

$$\implies Q = (H_1)^{-1} = H_1^{-1} \overset{ORTH}{=} H_1^T \overset{SYM}{=} H_1$$

Full QR Factorization via Householder Reflectors

Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$. Then:

$$\mathbf{a}_1 := (a_{11}, a_{21}, a_{31})^T \implies \mathbf{h}_1 = \mathbf{a}_1 \pm \overline{\text{sign}}(a_{11}) \cdot ||\mathbf{a}_1||_2 \cdot \hat{\mathbf{e}}_1 \implies \hat{\mathbf{h}}_1 = \mathbf{h}_1 / ||\mathbf{h}_1||_2$$

$$\implies H_1' = I - 2\hat{\mathbf{h}}_1 \hat{\mathbf{h}}_1^T \implies H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = H_1'$$

$$H_1 A = \begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}' \\ 0 & a_{32}' \end{bmatrix}$$

$$\mathbf{a}_1' := (a_{22}', a_{32}')^T \implies \mathbf{h}_2 = \mathbf{a}_1' \pm \overline{\text{sign}}(a_{22}') \cdot ||\mathbf{a}_1'||_2 \cdot \hat{\mathbf{e}}_1 \implies \hat{\mathbf{h}}_2 = \mathbf{h}_2 / ||\mathbf{h}_2||_2$$

$$\implies H_2' = I - 2\hat{\mathbf{h}}_2 \hat{\mathbf{h}}_2^T \implies H_2 = \begin{bmatrix} I_{1\times 1} & \\ & H_2' \end{bmatrix} = \begin{bmatrix} 1 & \\ & H_2' \end{bmatrix}$$

$$H_2 H_1 A = \begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}'' \\ 0 & 0 \end{bmatrix} = R$$

$$\implies Q = (H_2 H_1)^{-1} = H_1^{-1} H_2^{-1} \overset{ORTH}{=} H_1^T H_2^T \overset{SYM}{=} H_1 H_2$$

Full QR Factorization via Householder Reflectors

$$\text{Let} \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \implies H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = \begin{bmatrix} H_1' \end{bmatrix}$$

$$\implies H_1 A = \begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}' \end{bmatrix} = R$$

$$\implies Q = (H_1)^{-1} = H_1^{-1} \overset{ORTH}{=} H_1^T \overset{SYM}{=} H_1$$

Full QR Factorization via Householder Reflectors

Let $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \implies H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = \begin{bmatrix} H_1' \end{bmatrix}$$

$$\implies H_1 A = \begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}' \\ 0 & a_{32}' \end{bmatrix} \implies H_2 = \begin{bmatrix} I_{1\times 1} & \\ & H_2' \end{bmatrix} = \begin{bmatrix} 1 & \\ & H_2' \end{bmatrix}$$

$$\implies H_2 H_1 A = \begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}'' \\ 0 & 0 \end{bmatrix} = R$$

$$\implies Q = (H_2 H_1)^{-1} = H_1^{-1} H_2^{-1} \stackrel{ORTH}{=} H_1^T H_2^T \stackrel{SYM}{=} H_1 H_2$$

Full QR Factorization via Householder Reflectors

Let $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \implies H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = \begin{bmatrix} H_1' \end{bmatrix}$$

$$\implies H_1 A = \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \\ 0 & a_{42}' & a_{43}' \end{bmatrix} \implies H_2 = \begin{bmatrix} I_{1\times 1} & \\ & H_2' \end{bmatrix} = \begin{bmatrix} 1 & \\ & H_2' \end{bmatrix}$$

$$\implies H_2 H_1 A = \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ 0 & a_{22}'' & a_{23}'' \\ 0 & 0 & a_{33}'' \\ 0 & 0 & a_{43}'' \end{bmatrix} \implies H_3 = \begin{bmatrix} I_{2\times 2} & \\ & H_3' \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & H_3' \end{bmatrix}$$

$$\implies H_3 H_2 H_1 A = \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ 0 & a_{22}'' & a_{23}'' \\ 0 & 0 & a_{33}''' \\ 0 & 0 & 0 \end{bmatrix} = R$$

$$\implies Q = (H_3 H_2 H_1)^{-1} = H_1^{-1} H_2^{-1} H_3^{-1} \overset{ORTH}{=} H_1^T H_2^T H_3^T \overset{SYM}{=} H_1 H_2 H_3$$

Which Householder Reflector to Choose??

There are two Householder Reflectors to choose from:

$$\mathbf{h}_+ = \mathbf{a}_1^{(1)} - \overline{\text{sign}}(a_{11}^{(1)}) \cdot ||\mathbf{a}_1^{(1)}||_2 \cdot \hat{\mathbf{e}}_1$$
$$\mathbf{h}_- = \mathbf{a}_1^{(1)} + \overline{\text{sign}}(a_{11}^{(1)}) \cdot ||\mathbf{a}_1^{(1)}||_2 \cdot \hat{\mathbf{e}}_1$$

So, given the Householder Reflector expression containing a $\pm$ symbol:

$$\mathbf{h}_1 = \mathbf{a}_1^{(1)} \pm \overline{\text{sign}}(a_{11}^{(1)}) \cdot ||\mathbf{a}_1^{(1)}||_2 \cdot \hat{\mathbf{e}}_1$$

...which sign should one choose??

Answer: It depends!

If computing Householder Reflectors by hand (using exact arithmetic),
either $+$ or $-$ is fine,
so pick one.

However, if computing Householder Reflectors via a computer,
always pick the $+$ (i.e. always pick $\mathbf{h}_-$)
because the result will be more accurate.

S.J. Leon, *Linear Algebra with Applications*, 9^{th} Ed., Pearson, 2015.
A. Householder, "Unitary Triangularization of a Nonsymmetric Matrix", *J. ACM*, **5** (1958), 339-342.

Full QR Factorization via Householder Reflectors

Proposition

(Full QR Factorization via Householder Reflectors)

GIVEN: Tall or square $(m \geq n)$ full column rank matrix $A_{m\times n}$ with columns $\mathbf{a}_k$.
TASK: Factor $A = QR$ where $Q_{m\times m}$ has orthonormal columns $\widehat{\mathbf{q}}_k$ and $R_{m\times n}$ is upper triangular.

(0) For each upcoming $\pm$: by computer, always pick $+$; by hand, either $+$ or $-$ is fine, pick one.
(1) Build Householder Reflector, H_1, that nullifies sub-diagonal 1^{st} column of A:

$$\mathbf{a}_1^{(1)} := (a_{11}, \cdots, a_{m1})^T \implies \mathbf{h}_1 = \mathbf{a}_1^{(1)} \pm \overline{sign}(a_{11}^{(1)}) \cdot ||\mathbf{a}_1^{(1)}||_2 \cdot \hat{\mathbf{e}}_1 \implies \hat{\mathbf{h}}_1 = \mathbf{h}_1/||\mathbf{h}_1||_2$$
$$\implies H_1' = I - 2\hat{\mathbf{h}}_1\hat{\mathbf{h}}_1^T \implies H_1 = \begin{bmatrix} I_{0\times 0} & \\ & H_1' \end{bmatrix} = H_1'$$

(2) For each $j = 2, \cdots, n$:

Build Householder Reflector, H_j, that nullifies sub-diagonal j^{th} column of $H_jH_{j-1}\cdots H_2H_1A := C_j$:

$$\mathbf{c}_1^{(j)} := \left(c_{jj}^{(j)}, \cdots, c_{mj}^{(j)}\right)^T \implies \mathbf{h}_j = \mathbf{c}_1^{(j)} \pm \overline{sign}(c_{jj}^{(j)}) \cdot ||\mathbf{c}_1^{(j)}||_2 \cdot \hat{\mathbf{e}}_1 \implies \hat{\mathbf{h}}_j = \mathbf{h}_j/||\mathbf{h}_j||_2$$
$$\implies H_j' = I - 2\hat{\mathbf{h}}_j\hat{\mathbf{h}}_j^T \implies H_j = \begin{bmatrix} I_{(j-1)\times(j-1)} & \\ & H_j' \end{bmatrix}$$

(3) $R = H_nH_{n-1}\cdots H_2H_1A$

(4) $Q = H_1H_2\cdots H_{n-1}H_n$

S.J. Leon, *Linear Algebra with Applications*, 9^{th} Ed., Pearson, 2015.
A. Householder, "Unitary Triangularization of a Nonsymmetric Matrix", *J. ACM*, **5** (1958), 339-342.

Fin.