Singular Value Decomposition (SVD): $A = U\Sigma V^T$

Linear Algebra

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18 September 2022

PART I:

Symmetric Matrices (Revisited)
Orthogonal Matrices (Revisited)
Orthogonal Diagonalization (Revisited)

Symmetric Matrices (Definition)

The notion of a symmetric matrix is fundamental for later concepts & courses:

Definition

(Symmetric Matrix)

A square matrix $A \in \mathbb{R}^{n \times n}$ is **symmetric** if A is equal to its transpose: $A^T = A$

Corollary

(Diagonal Matrices are Symmetric)

A diagonal matrix $D \in \mathbb{R}^{n \times n}$ is symmetric.

Symmetric Matrices: $\begin{bmatrix} 1 & \mathbf{2} \\ \mathbf{2} & 3 \end{bmatrix}$, $\begin{bmatrix} -4 & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & 4 & \mathbf{5} \\ \mathbf{3} & \mathbf{5} & 6 \end{bmatrix}$

Not Symmetric: $\begin{bmatrix} -4 & \mathbf{0} \\ \mathbf{1} & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & \mathbf{2} & \mathbf{3} \\ \mathbf{9} & 4 & \mathbf{5} \\ \mathbf{8} & \mathbf{0} & 6 \end{bmatrix}$, $\begin{bmatrix} 1 & \mathbf{2} & \mathbf{3} \\ \mathbf{9} & 4 & \mathbf{5} \\ \mathbf{3} & \mathbf{5} & 6 \end{bmatrix}$

Symmetric Matrices (Properties)

Symmetric matrices have some very nice properties:

Theorem

(The Real Spectrum Theorem)

Let symmetric matrix $S \in \mathbb{R}^{n \times n}$. *Then the following all hold:*

- S *is diagonalizable*
- *All eigenvalues of* S *are real*
- *If some eigenvalue* λ_k *repeats, its multiplicities match:* *AM*$[\lambda_k] =$ *GM*$[\lambda_k]$
  i.e. If λ_k *occurs* j *times, then* λ_k *has* j *linearly independent eigenvectors:*

$$\mathbf{x}_{k,1},\ \mathbf{x}_{k,2},\ \ldots,\ \mathbf{x}_{k,j-1},\ \mathbf{x}_{k,j}$$

Eigenvectors of a Symmetric Matrix (Revisited)

Eigenvectors of distinct eigenvalues of a symmetric matrix have a benefit:

Theorem

(Eigenvectors of a Symmetric Matrix)

Let symmetric matrix $S \in \mathbb{R}^{n\times n}$ have eigenpairs $(\lambda_1, \mathbf{x}_1), (\lambda_2, \mathbf{x}_2)$. Then:

If eigenvalues λ_1, λ_2 are distinct, then eigenvectors $\mathbf{x}_1, \mathbf{x}_2$ are orthogonal.

i.e. If $\lambda_1 \neq \lambda_2$, then $\mathbf{x}_1 \perp \mathbf{x}_2$.

Orthogonal Matrices (Revisited)

Question: When is the inverse of a square matrix is simply its transpose??
Answer: When the square matrix is orthogonal:

Definition

(Orthogonal Matrix)

A square matrix $Q \in \mathbb{R}^{n \times n}$ is **orthogonal** if Q is invertible and $Q^{-1} = Q^T$

Corollary

(Determining if a Matrix is Orthogonal)

A square matrix $Q \in \mathbb{R}^{n \times n}$ is ***orthogonal*** $\iff Q^T Q = I$

Theorem

(An Orthogonal Matrix has Orthonormal Columns)

A square matrix Q is ***orthogonal*** $\iff$ *its columns form an orthonormal set.*

Orthogonally Diagonalizable Matrices (Revisited)

Definition

(Orthogonally Diagonalizable Matrix)

Let square matrix $A \in \mathbb{R}^{n \times n}$.
Then A is **orthogonally diagonalizable** if $\exists Q \in \mathbb{R}^{n \times n}$ s.t. Q is orthogonal and

$$Q^T A Q = D \text{ where } D \text{ is diagonal.}$$

Theorem

(Symmetric Matrices are Orthogonally Diagonalizable)

Square matrix A is orthgonally diagonalizable $\iff$ A is symmetric.

PART II:

Full Singular Value Decomposition (SVD)

The Four Fundamental Matrix Subspaces via Full SVD

Reduced Singular Value Decomposition (SVD)

Full SVD (Motivation)

Recall that only symmetric matrices have an ortho-eigen-decomposition:

$$S_{n\times n} = Q\Lambda Q^T$$

Recall further that eigenvalues of a symmetric matrix are all real.
Recall further that eigenvalues of a square matrix may be complex in general.

Question:

If this decomposition is relaxed by allowing two unalike orthogonal matrices...
...but forcing all the middle's main diagonal entries to be real numbers...
...can such a decomposition exist for all rectangular matrices???

i.e. For all $m \times n$ real-entried matrices A, is this possible:

$A_{m\times n} = U\Sigma V^T$ s.t. $U^TU = I_{m\times m}$, $V^TV = I_{n\times n}$, $\Sigma_{m\times n}$ is real-entried diagonal

Answer: Turns out, YES! Let's derive it!

Full SVD (Motivation)

Given: $m \times n$ rectangular matrix A such that $\text{rank}(A) = r$.

Question: Is it possible for Full SVD to recycle a prior decomposition?

Answer: YES! The ortho-eigen-decomposition for symmetric matrices:

$$S_{n\times n} = Q\Lambda Q^T$$

Question: How to produce a square symmetric matrix involving only matrix A?

Answer: A^TA and AA^T

Full SVD (Properties of A^TA & AA^T)

Theorem

(Properties of A^TA & AA^T)

Let A be a tall or square $m \times n$ matrix. Then:
(i) A^TA *and* AA^T *are both symmetric.*
(ii) *The eigenvalues of* A^TA *and* AA^T *are all real.*
(iii) *The eigenvectors of* A^TA *and* AA^T *are orthonormal.*
(iv) $(\mu, \mathbf{v}) \neq (0, \vec{\mathbf{0}})$ *is an eigenpair of* $A^TA \iff (\mu, A\mathbf{v})$ *is an eigenpair of* AA^T*.*
(v) *The eigenvalues of* A^TA *and* AA^T *are non-negative.*
(vi) *NulSp*$(A^TA) =$ *NulSp*(A), *NulSp*$(AA^T) =$ *NulSp*(A^T)
(vii) *ColSp*$(A^TA) =$ *ColSp*(A^T), *ColSp*$(AA^T) =$ *ColSp*(A)

Full SVD (Properties of A^TA & AA^T)

Theorem

(Properties of A^TA & AA^T)

Let A be a tall or square $m \times n$ matrix. Then:
(i) A^TA *and* AA^T *are both symmetric.*
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(iv) $(\mu, \mathbf{v}) \neq (0, \vec{\mathbf{0}})$ *is an eigenpair of* $A^TA \iff (\mu, A\mathbf{v})$ *is an eigenpair of* AA^T*.*
(v) *The eigenvalues of* A^TA *and* AA^T *are non-negative.*
(vi) *NulSp*$(A^TA) =$ *NulSp*(A), *NulSp*$(AA^T) =$ *NulSp*(A^T)
(vii) *ColSp*$(A^TA) =$ *ColSp*(A^T), *ColSp*$(AA^T) =$ *ColSp*(A)

PROOF: (i) $(A^TA)^T = A^TA^{TT} = A^TA \implies A^TA$ is symmetric, AA^T is left to the reader to prove.

(ii) & (iii) Both follow immediately since AA^T and AA^T are symmetric matrices.

(iv) $A^TA\mathbf{v} \overset{EIG}{=} \mu\mathbf{v} \overset{A\cdot}{\iff} A(A^TA\mathbf{v}) = A(\mu\mathbf{v}) \overset{ASSOC.}{\iff} (AA^T)(A\mathbf{v}) = A(\mu\mathbf{v})$
$\overset{FACTOR}{\iff} (AA^T)(A\mathbf{v}) = \mu(A\mathbf{v}) \overset{EIG}{\iff} (\mu, A\mathbf{v})$ is an eigenpair of AA^T.

(v) Let $(\mu, \mathbf{v})$ be an eigenpair of A^TA and $(\mu, \mathbf{u})$ be an eigenpair of AA^T. Then:

$$||A\mathbf{v}||_2^2 = (A\mathbf{v})^T(A\mathbf{v}) = \mathbf{v}^TA^TA\mathbf{v} \overset{EIG}{=} \mathbf{v}^T\mu\mathbf{v} = \mu\mathbf{v}^T\mathbf{v} = \mu||\mathbf{v}||_2^2 \implies \mu = \frac{||A\mathbf{v}||_2^2}{||\mathbf{v}||_2^2} \geq 0$$

$$||A^T\mathbf{u}||_2^2 = (A^T\mathbf{u})^T(A^T\mathbf{u}) = \mathbf{u}^TAA^T\mathbf{u} \overset{EIG}{=} \mathbf{u}^T\mu\mathbf{u} = \mu\mathbf{u}^T\mathbf{u} = \mu||\mathbf{u}||_2^2 \implies \mu = \frac{||A^T\mathbf{u}||_2^2}{||\mathbf{u}||_2^2} \geq 0 \quad \square$$

Full SVD (Properties of A^TA & AA^T)

Theorem

(Properties of A^TA & AA^T)

Let A be a tall or square $m \times n$ matrix. Then:
(i) A^TA and AA^T are both symmetric.
(ii) *The eigenvalues of A^TA and AA^T are all real.*
(iii) *The eigenvectors of A^TA and AA^T are orthonormal.*
(iv) $(\mu, \mathbf{v}) \neq (0, \vec{\mathbf{0}})$ *is an eigenpair of A^TA* $\iff$ $(\mu, A\mathbf{v})$ *is an eigenpair of AA^T.*
(v) *The eigenvalues of A^TA and AA^T are non-negative.*
(vi) *NulSp*$(A^TA) =$ *NulSp*(A), *NulSp*$(AA^T) =$ *NulSp*(A^T)
(vii) *ColSp*$(A^TA) =$ *ColSp*(A^T), *ColSp*$(AA^T) =$ *ColSp*(A)

PROOF:

$$(vi)\ \text{NulSp}(A^TA) := \left\{\mathbf{x} \in \mathbb{R}^n : A^TA\mathbf{x} = \vec{\mathbf{0}}\right\} = \left\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \vec{\mathbf{0}}\right\} := \text{NulSp}(A)$$

$$\text{NulSp}(AA^T) := \left\{\mathbf{y} \in \mathbb{R}^m : AA^T\mathbf{y} = \vec{\mathbf{0}}\right\} = \left\{\mathbf{y} \in \mathbb{R}^m : A^T\mathbf{y} = \vec{\mathbf{0}}\right\} := \text{NulSp}(A^T)$$

$$(vii)\ \text{ColSp}(A^TA) \overset{FTLA}{=} \text{NulSp}(A^TA)^\perp \overset{(vi)}{=} \text{NulSp}(A)^\perp \overset{FTLA}{=} \text{ColSp}(A^T)$$

$$\text{ColSp}(AA^T) \overset{FTLA}{=} \text{NulSp}(AA^T)^\perp \overset{(vi)}{=} \text{NulSp}(A^T)^\perp \overset{FTLA}{=} \text{ColSp}(A)$$

Full SVD (Derivation, starting from A^TA)

Given: $m \times n$ rectangular matrix A such that $\text{rank}(A) = r$.

$$\text{Factor } A^TA = VMV^T \text{ s.t. } \left\{ \begin{array}{ccc} V^TV = VV^T = I_{n\times n} & , & M = \text{diag}(\mu_1, \cdots, \mu_n) \\ \langle \hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j \rangle_2 = \delta_{ij} & , & A^TA\hat{\mathbf{v}}_k = \mu_k \hat{\mathbf{v}}_k \ \forall k \leq n \end{array} \right\} \text{ Then:}$$

$$||A\hat{\mathbf{v}}_k||_2^2 = (A\hat{\mathbf{v}}_k)^T(A\hat{\mathbf{v}}_k) \stackrel{T}{=} \hat{\mathbf{v}}_k^T(A^TA)\hat{\mathbf{v}}_k \stackrel{EIG}{=} \hat{\mathbf{v}}_k^T\mu_k\hat{\mathbf{v}}_k = \mu_k(\hat{\mathbf{v}}_k^T\hat{\mathbf{v}}_k) = \mu_k||\hat{\mathbf{v}}_k||_2^2 = \mu_k \cdot 1 = \mu_k$$

$$||A\hat{\mathbf{v}}_k||_2 = \sqrt{\mu_k} := \sigma_k \implies \hat{\mathbf{u}}_k := A\hat{\mathbf{v}}_k/\sigma_k$$

Descend-sort-label the **singular values** of A like so: $\sigma_1 \geq \cdots \geq \sigma_r > 0$

$$\langle \hat{\mathbf{u}}_i, \hat{\mathbf{u}}_j \rangle_2 = \hat{\mathbf{u}}_i^T\hat{\mathbf{u}}_j = \left(\frac{A\hat{\mathbf{v}}_i}{\sigma_i}\right)^T \left(\frac{A\hat{\mathbf{v}}_j}{\sigma_j}\right) \stackrel{T}{=} \frac{\hat{\mathbf{v}}_i^TA^TA\hat{\mathbf{v}}_j}{\sigma_i\sigma_j} \stackrel{EIG}{=} \frac{\mu_j}{\sigma_i\sigma_j} \cdot \hat{\mathbf{v}}_i^T\hat{\mathbf{v}}_j \stackrel{\perp}{=} \delta_{ij} \implies \langle \hat{\mathbf{u}}_i, \hat{\mathbf{u}}_j \rangle_2 = \delta_{ij}$$

$$\therefore \left\{ \begin{array}{cccl} A\hat{\mathbf{v}}_k & = & \sigma_k\hat{\mathbf{u}}_k & \text{for } \ 1 \leq k \leq r \\ A\hat{\mathbf{v}}_k & = & \vec{\mathbf{0}} & \text{for } \ r < k \leq n \end{array} \right.$$

Note that $\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_n$ are called the **right-singular vectors** of A

$$\therefore AV = U\Sigma \implies \boxed{A = U\Sigma V^T}$$

Full SVD (Derivation, starting from AA^T)

Given: $m \times n$ rectangular matrix A such that rank$(A) = r$.

$$\text{Factor } AA^T = UMU^T \text{ s.t. } \left\{ \begin{array}{ccc} U^T U = UU^T = I_{m\times m} & , & M = \text{diag}(\mu_1, \cdots, \mu_m) \\ \langle \hat{\mathbf{u}}_i, \hat{\mathbf{u}}_j \rangle_2 = \delta_{ij} & , & AA^T \hat{\mathbf{u}}_k = \mu_k \hat{\mathbf{u}}_k \ \forall k \leq m \end{array} \right\} \text{ Then:}$$

$$||A^T \hat{\mathbf{u}}_k||_2^2 = (A^T \hat{\mathbf{u}}_k)^T (A^T \hat{\mathbf{u}}_k) \overset{T}{=} \hat{\mathbf{u}}_k^T (AA^T) \hat{\mathbf{u}}_k \overset{EIG}{=} \hat{\mathbf{u}}_k^T \mu_k \hat{\mathbf{u}}_k = \mu_k (\hat{\mathbf{u}}_k^T \hat{\mathbf{u}}_k) = \mu_k ||\hat{\mathbf{u}}_k||_2^2 = \mu_k \cdot 1 = \mu_k$$

$$||A^T \hat{\mathbf{u}}_k||_2 = \sqrt{\mu_k} := \sigma_k \implies \hat{\mathbf{v}}_k := A^T \hat{\mathbf{u}}_k / \sigma_k$$

Descend-sort-label the **singular values** of A like so: $\sigma_1 \geq \cdots \geq \sigma_r > 0$

$$\langle \hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j \rangle_2 = \hat{\mathbf{v}}_i^T \hat{\mathbf{v}}_j = \left(\frac{A^T \hat{\mathbf{u}}_i}{\sigma_i} \right)^T \left(\frac{A^T \hat{\mathbf{u}}_j}{\sigma_j} \right) \overset{T}{=} \frac{\hat{\mathbf{u}}_i^T AA^T \hat{\mathbf{u}}_j}{\sigma_i \sigma_j} \overset{EIG}{=} \frac{\mu_j}{\sigma_i \sigma_j} \cdot \hat{\mathbf{u}}_i^T \hat{\mathbf{u}}_j \overset{\perp}{=} \delta_{ij} \implies \langle \hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j \rangle_2 = \delta_{ij}$$

$$\therefore \left\{ \begin{array}{ccccc} A^T \hat{\mathbf{u}}_k & = & \sigma_k \hat{\mathbf{v}}_k & \text{for} & 1 \leq k \leq r \\ A^T \hat{\mathbf{u}}_k & = & \vec{\mathbf{0}} & \text{for} & r < k \leq m \end{array} \right.$$

Note that $\hat{\mathbf{u}}_1, \cdots, \hat{\mathbf{u}}_m$ are called the **left-singular vectors** of A

$$\therefore A^T U = V\Sigma^T \implies A^T = V\Sigma^T U^T \overset{T}{\implies} \boxed{A = U\Sigma V^T}$$

Full SVD (Derivation, final matrix shapes)

$$\text{rank}(A) = r$$

$$A = U\Sigma V^T$$

$$A_{n\times n} = \underbrace{\begin{bmatrix} | & & | \\ \hat{\mathbf{u}}_1 & \cdots & \hat{\mathbf{u}}_n \\ | & & | \end{bmatrix}}_{n\times n} \underbrace{\left[\begin{array}{c|c} \dot{\Sigma}_{r\times r} & \\ \hline & O_{(n-r)\times(n-r)} \end{array}\right]}_{n\times n} \underbrace{\begin{bmatrix} \text{---} & \hat{\mathbf{v}}_1 & \text{---} \\ & \vdots & \\ \text{---} & \hat{\mathbf{v}}_n & \text{---} \end{bmatrix}}_{n\times n}$$

$$A_{m\times n} = \underbrace{\begin{bmatrix} | & & | \\ \hat{\mathbf{u}}_1 & \cdots & \hat{\mathbf{u}}_m \\ | & & | \end{bmatrix}}_{m\times m} \underbrace{\left[\begin{array}{c|c} \dot{\Sigma}_{r\times r} & \\ \hline & O_{(m-r)\times(n-r)} \end{array}\right]}_{m\times n} \underbrace{\begin{bmatrix} \text{---} & \hat{\mathbf{v}}_1 & \text{---} \\ & \vdots & \\ \text{---} & \hat{\mathbf{v}}_n & \text{---} \end{bmatrix}}_{n\times n}$$

where: $U^{-1} = U^T, \;\; V^{-1} = V^T, \;\; \dot{\Sigma} := \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$ with $\sigma_1 \geq \cdots \geq \sigma_r > 0$

Full SVD (Procedure)

Proposition

(Full SVD)

GIVEN: $m \times n$ tall $(m \geq n)$ or square $(m = n)$ matrix A with column rank $r \leq n$:

TASK: Factor $A = U\Sigma V^T$ where: $U^T U = I_{m\times m}$, $V^T V = I_{n\times n}$, Σ is $m \times n$ diagonal

1. Find the n eigenvalues $\mu_1 \geq \cdots \geq \mu_r > \mu_{r+1} = \cdots = \mu_n = 0$ of symmetric matrix $A^T A$.
2. Find the r **right-singular vectors** of A: $\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_r =$ eigenvectors of $A^T A$ w.r.t. $\mu_1, \cdots, \mu_r$
3. Find the r **singular values** of A: $\sigma_1 := \sqrt{\mu_1}, \cdots, \sigma_r := \sqrt{\mu_r}$
4. Find the r **left-singular vectors** of A: $\hat{\mathbf{u}}_1 := A\hat{\mathbf{v}}_1/\sigma_1, \cdots, \hat{\mathbf{u}}_r := A\hat{\mathbf{v}}_r/\sigma_r$
5. If $r < n$, use Gram-Schmidt on std basis vectors $\hat{\mathbf{e}}_{r+1}, \cdots, \hat{\mathbf{e}}_n \in \mathbb{R}^n$ to find: $\hat{\mathbf{v}}_{r+1}, \cdots, \hat{\mathbf{v}}_n$
6. If $r < m$, use Gram-Schmidt on std basis vectors $\hat{\mathbf{e}}_{r+1}, \cdots, \hat{\mathbf{e}}_m \in \mathbb{R}^m$ to find: $\hat{\mathbf{u}}_{r+1}, \cdots, \hat{\mathbf{u}}_m$
7. Form the matrices comprising the Full SVD like so:

$$A_{m\times n} = \underbrace{\begin{bmatrix} | & & | \\ \hat{\mathbf{u}}_1 & \cdots & \hat{\mathbf{u}}_m \\ | & & | \end{bmatrix}}_{m\times m} \underbrace{\left[\begin{array}{c|c} \dot{\Sigma}_{r\times r} & \\ \hline & O_{(m-r)\times(n-r)} \end{array}\right]}_{m\times n} \underbrace{\begin{bmatrix} \text{---} & \hat{\mathbf{v}}_1 & \text{---} \\ & \vdots & \\ \text{---} & \hat{\mathbf{v}}_n & \text{---} \end{bmatrix}}_{n\times n} \equiv U\Sigma V^T$$

where: $O_{(m-r)\times(n-r)}$ is zero matrix and $\dot{\Sigma} :=$ diag$(\sigma_1, \cdots, \sigma_r)$ with $\sigma_1 \geq \cdots \geq \sigma_r > 0$

The Four Fundamental Matrix Subspaces via Full SVD

Given: $m \times n$ rectangular matrix A s.t. rank$(A) = r$ and $A = U\Sigma V^T$. Then:

$$\begin{cases} A\hat{\mathbf{v}}_k = \sigma_k \hat{\mathbf{u}}_k, \; k \le r \\ A\hat{\mathbf{v}}_k = \vec{\mathbf{0}}, \; r < k \end{cases} \implies \begin{array}{l} \text{ColSp}(A) := \text{Span}\{A\hat{\mathbf{v}}_k : k \le n\} = \text{Span}\{\hat{\mathbf{u}}_1, \cdots, \hat{\mathbf{u}}_r\} \\ \text{NulSp}(A) := \text{Span}\{\mathbf{x} : A\mathbf{x} = \vec{\mathbf{0}}\} = \text{Span}\{\hat{\mathbf{v}}_{r+1}, \cdots, \hat{\mathbf{v}}_n\} \end{array}$$

$$\begin{cases} A^T\hat{\mathbf{u}}_k = \sigma_k \hat{\mathbf{v}}_k, \; k \le r \\ A^T\hat{\mathbf{u}}_k = \vec{\mathbf{0}}, \; r < k \end{cases} \implies \begin{array}{l} \text{ColSp}(A^T) := \text{Span}\{A^T\hat{\mathbf{u}}_k : k \le m\} = \text{Span}\{\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_r\} \\ \text{NulSp}(A^T) := \text{Span}\{\mathbf{y} : A^T\mathbf{y} = \vec{\mathbf{0}}\} = \text{Span}\{\hat{\mathbf{u}}_{r+1}, \cdots, \hat{\mathbf{u}}_m\} \end{array}$$

The Four Fundamental Matrix Subspaces via Full SVD

Theorem

(The Four Fundamental Matrix Subspaces via Full SVD)

GIVEN: $m \times n$ *tall/square* $(m \geq n)$ *matrix* A *with column rank* $r \leq n$ *and Full SVD* $A = U\Sigma V^T$*.*
Then, the four fundamental matrix subspaces of A *are (directly via the Full SVD):*

- *ColSp*$(A) =$ *Span*$\{\hat{\mathbf{u}}_1, \cdots, \hat{\mathbf{u}}_r\}$
- *NulSp*$(A) =$ *Span*$\{\hat{\mathbf{v}}_{r+1}, \cdots, \hat{\mathbf{v}}_n\}$
- *ColSp*$(A^T) =$ *Span*$\{\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_r\}$
- *NulSp*$(A^T) =$ *Span*$\{\hat{\mathbf{u}}_{r+1}, \cdots, \hat{\mathbf{u}}_m\}$

Moreover, the relations among these fundamental subspaces are now immediate:

$$\begin{array}{cc} (i) & \text{dim ColSp}(A) = r \\ (ii) & \text{dim ColSp}(A^T) = r \end{array}$$

$$\begin{array}{cc} (iii) & \mathbb{R}^m = \text{ColSp}(A) \oplus \text{NulSp}(A^T) \\ (iv) & \mathbb{R}^n = \text{ColSp}(A^T) \oplus \text{NulSp}(A) \end{array}$$

Reduced SVD (Procedure)

Proposition

(Reduced SVD)

GIVEN: $m \times n$ tall $(m \geq n)$ or square $(m = n)$ matrix A with column rank $r \leq n$:

TASK: Factor $A = \hat{U}\dot{\Sigma}\hat{V}^T$ where: $\hat{U}^T\hat{U} = I_{m\times m}$, $\hat{V}^T\hat{V} = I_{n\times n}$, $\dot{\Sigma}$ is $r \times r$ diagonal

1. *Find the n eigenvalues $\mu_1 \geq \cdots \geq \mu_r > \mu_{r+1} = \cdots = \mu_n = 0$ of symmetric matrix $A^T A$.*
2. *Find the r **right-singular vectors** of A: $\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_r =$ eigenvectors of $A^T A$ w.r.t. $\mu_1, \cdots, \mu_r$*
3. *Find the r **singular values** of A: $\sigma_1 := \sqrt{\mu_1}, \cdots, \sigma_r := \sqrt{\mu_r}$*
4. *Find the r **left-singular vectors** of A: $\hat{\mathbf{u}}_1 := A\hat{\mathbf{v}}_1/\sigma_1, \cdots, \hat{\mathbf{u}}_r := A\hat{\mathbf{v}}_r/\sigma_r$*
5. *Form the matrices comprising the Reduced SVD like so:*

$$A_{m\times n} = \underbrace{\begin{bmatrix} | & & | \\ \hat{\mathbf{u}}_1 & \cdots & \hat{\mathbf{u}}_r \\ | & & | \end{bmatrix}}_{m\times r} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}}_{r\times r} \underbrace{\begin{bmatrix} \text{---} & \hat{\mathbf{v}}_1 & \text{---} \\ & \vdots & \\ \text{---} & \hat{\mathbf{v}}_r & \text{---} \end{bmatrix}}_{r\times n} \equiv \hat{U}\dot{\Sigma}\hat{V}^T$$

where: $\sigma_1 \geq \cdots \geq \sigma_r > 0$

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Fin

Fin.