DOMAIN OF FUNCTIONS

FUNDAMENTAL DEFINITIONS:

- The **domain** of a function f, denoted $\text{Dom}(f)$, is the set of all real numbers x such that $f(x)$ is defined.
- The above domain definition in set notation: $\text{Dom}(f) := \{x \in \mathbb{R} : f(x) \text{ is defined}\}$
- The **range** of a function f, denoted $\text{Rng}(f)$, is the set of all real numbers y such that $y = f(x) \;\; \forall x \in \text{Dom}(f)$
- The above range definition in set notation: $\text{Rng}(f) := \{y \in \mathbb{R} : y = f(x) \;\; \forall x \in \text{Dom}(f)\}$
- In general, finding $\text{Rng}(f)$ is very tough without graphing f or using advanced analysis (beyond the scope of Calculus I).

DOMAIN OF SIMPLE FUNCTIONS:

- (DM.1) $f_1 \in \{1, x, x^2, x^3, x^4, x^5, \ldots\} \implies \text{Dom}(f_1) = \mathbb{R}$
- (DM.2) $f_2 \in \{\sqrt{x}, \sqrt[4]{x}, \sqrt[6]{x}, \sqrt[8]{x}, \ldots\} \implies [\, \text{Dom}(f_2) = [0, \infty) \iff x \geq 0]$
- (DM.3) $f_3 \in \{\sqrt[3]{x}, \sqrt[5]{x}, \sqrt[7]{x}, \sqrt[9]{x}, \ldots\} \implies \text{Dom}(f_3) = \mathbb{R}$
- (DM.4) $f_4 \in \left\{\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}, \frac{1}{x^5}, \frac{1}{x^6}, \cdots\right\} \implies [\, \text{Dom}(f_4) = \mathbb{R} \setminus \{0\} \iff x \neq 0]$
- (DM.5) $f_5 \in \left\{\frac{1}{\sqrt{x}}, \frac{1}{\sqrt[4]{x}}, \frac{1}{\sqrt[6]{x}}, \frac{1}{\sqrt[8]{x}}, \cdots\right\} \implies [\, \text{Dom}(f_5) = \mathbb{R}_+ = (0, \infty) \iff x > 0]$
- (DM.6) $f_6 \in \left\{\frac{1}{\sqrt[3]{x}}, \frac{1}{\sqrt[5]{x}}, \frac{1}{\sqrt[7]{x}}, \frac{1}{\sqrt[9]{x}}, \cdots\right\} \implies \text{Dom}(f_6) = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$
- (DM.7) $f_7 \in \{e^x, (-2)^x, 3^x, (\sqrt{5})^x, \ldots\} \implies \text{Dom}(f_7) = \mathbb{R}$
- (DM.8) $f_8 \in \{\ln x, \log x, \log_b x, \ldots\} \implies \text{Dom}(f_8) = \mathbb{R}_+ = (0, \infty)$
- (DM.9) $f_9 \in \left\{\frac{1}{\ln x}, \frac{1}{\log x}, \frac{1}{\log_b x}, \cdots\right\} \implies \text{Dom}(f_9) = \mathbb{R}_+ \setminus \{1\} = (0, 1) \cup (1, \infty)$
- (DM.10) $f_{10} \in \{|x|, \sin x, \cos x\} \implies \text{Dom}(f_{10}) = \mathbb{R}$
- (DM.11) $f_{11} \in \{\tan x, \sec x\} \implies \left[\text{Dom}(f_{11}) = \mathbb{R} \setminus \{\cos x = 0\} \iff x \notin \left\{\cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots\right\}\right]$
- (DM.12) $f_{12} \in \{\cot x, \csc x\} \implies \left[\text{Dom}(f_{12}) = \mathbb{R} \setminus \{\sin x = 0\} \iff x \notin \{\ldots, -4\pi, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi, \ldots\}\right]$

DOMAIN OF ARITHMETIC COMBINATIONS OF FUNCTIONS:

- (DM.A) $\alpha \in \mathbb{R} \implies \text{Dom}(\alpha f) = \text{Dom}(f)$
- (DM.S) $\text{Dom}(f + g) = \text{Dom}(f) \cap \text{Dom}(g)$
- (DM.D) $\text{Dom}(f - g) = \text{Dom}(f) \cap \text{Dom}(g)$
- (DM.P) $\text{Dom}(fg) = \text{Dom}(f) \cap \text{Dom}(g)$
- (DM.Q) $\text{Dom}(f/g) = [\text{Dom}(f) \cap \text{Dom}(g)] \setminus \{g(x) = 0\}$ [i.e. Find $\text{Dom}(f) \cap \text{Dom}(g)$, then exclude x's where $g(x) = 0$]
- (DM.C) $\text{Dom}(f \circ g) = \{x \in \text{Dom}(g) : g(x) \in \text{Dom}(f)\}$ (**Composition** of f and g – In general, $f \circ g \neq g \circ f$)
- (DM.V) $\text{Dom}(f^{-1}) = \text{Rng}(f)$ ($f^{-1} \equiv$ **inverse** of f)

RANGE OF SIMPLE FUNCTIONS:

- (RG.1) $g_1 \in \{x, x^3, x^5, x^7, \sqrt[3]{x}, \sqrt[5]{x}, \tan x, \cot x, \ln x, \log x, \log_b x, \ldots\} \implies \text{Rng}(g_1) = \mathbb{R}$
- (RG.2) $g_2 \in \{|x|, x^2, x^4, x^6, \sqrt{x}, \sqrt[4]{x}, \sqrt[6]{x}, \ldots\} \implies \text{Rng}(g_2) = [0, \infty)$
- (RG.3) $g_3 \in \left\{\frac{1}{\sqrt{x}}, \frac{1}{\sqrt[3]{x}}, \frac{1}{\sqrt[4]{x}}, e^x, 3^x, (\sqrt{5})^x, \cdots\right\} \implies \text{Rng}(g_3) = (0, \infty)$
- (RG.4) $g_4 \in \{\sin x, \cos x\} \implies \text{Rng}(g_4) = [-1, 1]$
- (RG.5) $g_5 \in \{\csc x, \sec x\} \implies \text{Rng}(g_5) = \mathbb{R} \setminus (-1, 1) = (-\infty, -1] \cup [1, \infty)$
- (RG.V) $\text{Rng}(f^{-1}) = \text{Dom}(f)$ ($f^{-1} \equiv$ **inverse** of f)

EXAMPLE: Find the domain of $f(x) = 2x^3 - x + 10$

(Easy Method) Observe that f is a **polynomial**, and further observe that a polynomial is defined everywhere.

Hence, $\boxed{\text{Dom}(f) = \mathbb{R}}$

EXAMPLE: Find the domain of $f(x) = \dfrac{4}{2x - 10}$

(Easy Method) Observe that f is a **rational function** and is defined everywhere except where the denominator is zero.

$$\implies 2x - 10 = 0 \implies 2x = 10 \implies x = 5 \iff x \in \{5\}$$

Hence, $\boxed{\text{Dom}(f) = \mathbb{R} \setminus \{5\} = (-\infty, 5) \cup (5, \infty)}$

(General Method) Let $f_1(x) = 4, f_2(x) = 2x - 10$. Then, $f(x) = \dfrac{f_1(x)}{f_2(x)}$.

$$\implies \text{Dom}(f) = \text{Dom}\left(\frac{f_1}{f_2}\right) \overset{DM.Q}{=} \Big[\text{Dom}(f_1) \cap \text{Dom}(f_2)\Big] \setminus \{f_2(x) = 0\} \overset{DM.1}{=} \Big[\mathbb{R} \cap \mathbb{R}\Big] \setminus \{5\} = \mathbb{R} \setminus \{5\}$$

Hence, $\boxed{\text{Dom}(f) = \mathbb{R} \setminus \{5\} = (-\infty, 5) \cup (5, \infty)}$

EXAMPLE: Find the domain of $f(x) = \dfrac{x}{1 + x^4}$

(Easy Method) Observe that f is a **rational function** and is defined everywhere except where $1 + x^4 = 0$.

But, the equation $1 + x^4 = 0$ has no (real) solutions $\iff x \in \emptyset$. Hence, $\boxed{\text{Dom}(f) = \mathbb{R}}$

(General Method) Let $f_1(x) = 1, f_2(x) = 1 + x^4$. Then, $f(x) = \dfrac{f_1(x)}{f_2(x)}$.

$$\implies \text{Dom}(f) = \text{Dom}\left(\frac{f_1}{f_2}\right) \overset{DM.Q}{=} \Big[\text{Dom}(f_1) \cap \text{Dom}(f_2)\Big] \setminus \{f_2(x) = 0\} \overset{DM.1}{=} \Big[\mathbb{R} \cap \mathbb{R}\Big] \setminus \emptyset = \mathbb{R}.$$

Hence, $\boxed{\text{Dom}(f) = \mathbb{R}}$

EXAMPLE: Find the domain of $f(x) = x \ln x - \sqrt{x}$

(General Method) Let $f_1(x) = x, f_2(x) = \ln x, f_3 = \sqrt{x}$. Then, $f(x) = f_1(x)f_2(x) - f_3(x)$.

$$\implies \text{Dom}(f) = \text{Dom}(f_1 f_2 - f_3) = \Big[\text{Dom}(f_1) \cap \text{Dom}(f_2)\Big] \cap \text{Dom}(f_3) = \Big[\mathbb{R} \cap \mathbb{R}_+\Big] \cap [0, \infty) = \mathbb{R}_+ \cap [0, \infty) = \mathbb{R}_+$$

Hence, $\boxed{\text{Dom}(f) = \mathbb{R}_+ = (0, \infty)}$

EXAMPLE: Find the domain of $f(x) = \sqrt{10 - x}$

(General Method) Let $f_1(x) = 10 - x, f_2(x) = \sqrt{x}$. Then, $f = f_2 \circ f_1$.

$$\implies \text{Dom}(f) = \text{Dom}(f_2 \circ f_1) \overset{DM.C}{=} \{x \in \text{Dom}(f_1) : f_1(x) \in \text{Dom}(f_2)\} = \{x \in \mathbb{R} : (10 - x) \in [0, \infty)\}$$

$$= \{x \in \mathbb{R} : 10 - x \geq 0\} = \{x \in \mathbb{R} : x \leq 10\} = (-\infty, 10]$$

Hence, $\boxed{\text{Dom}(f) = (-\infty, 10]}$

EXAMPLE: Find the domain of $f(x) = \arcsin x$

(General Method) $\text{Dom}(f) = \text{Dom}(\arcsin x) \overset{DM.V}{=} \text{Rng}(\sin x) \overset{RG.4}{=} [-1, 1]$

Hence, $\boxed{\text{Dom}(f) = [-1, 1]}$