FUNCTIONS OF TWO VARIABLES: RELATIVE EXTREMA [SST 11.7]

- **OPEN, CLOSED, AND BOUNDED SETS IN $\mathbb{R}^2$:**
  - A set $S \subseteq \mathbb{R}^2$ is **open** if S contains none of its boundary.
  - A set $S \subseteq \mathbb{R}^2$ is **closed** if S contains all of its boundary.
  - $\mathbb{R}^2$ and $\emptyset$ (empty set) are **both open and closed**.
  - **Open disk** centered at (x_0, y_0) with radius $r > 0$: $\mathbb{D}(x_0, y_0; r) := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$
  - **Closed disk** centered at (x_0, y_0) with radius $r > 0$: $\overline{\mathbb{D}}(x_0, y_0; r) := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\}$
  - A set $S \subset \mathbb{R}^2$ is **bounded** if S is contained in an open disk.

- **SUFFICIENT CONDITION FOR EQUALITY OF MIXED 2^{nd}-ORDER PARTIALS:**
  - Let $f(x, y) \in C^{(2,2)}$. Then $f_{xy} = f_{yx}$

- **CRITICAL POINTS:**
  - Let $f(x, y)$ be defined on an open set $S \subseteq \mathbb{R}^2$ such that $(x_0, y_0) \in S$.

    Then (x_0, y_0) is a **critical point** of f is either one of the following is true:

    (i) $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

    (ii) At least one of $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ DNE

- **RELATIVE MIN's, RELATIVE MAX's, SADDLE POINTS (”FIRST PRINCIPLES” DEFINITIONS):**
  - Let $f(x, y)$ be defined on an open set $S \subseteq \mathbb{R}^2$ such that $(x_0, y_0) \in S$. Then:

    (x_0, y_0) is a **relative maximum** if $f(x, y) \leq f(x_0, y_0)$ $\forall (x, y) \in \mathbb{D}(x_0, y_0; r)$.

    (x_0, y_0) is a **relative minimum** if $f(x, y) \geq f(x_0, y_0)$ $\forall (x, y) \in \mathbb{D}(x_0, y_0; r)$.

    (x_0, y_0) is a **saddle point** if $\exists (x_1, y_1), (x_2, y_2) \in \mathbb{D}(x_0, y_0; r)$ s.t. $f(x_1, y_1) > f(x_0, y_0)$ and $f(x_2, y_2) < f(x_0, y_0)$.

- **RELATIVE MIN's, RELATIVE MAX's, SADDLE POINTS (2^{nd}-ORDER PARTIALS TEST):**
  - Let $f(x, y) \in C^{(2,2)}\Big(\mathbb{D}(x_0, y_0; r)\Big)$ s.t. f has a critical point at (x_0, y_0).

    Form the **discriminant** of f: $\Delta(x, y) := \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = f_{xx} f_{yy} - (f_{xy})^2$

    Then:

    (x_0, y_0) is a **relative max** if $\Big(\Delta(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0\Big)$ OR $\Big(\Delta(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) < 0\Big)$

    (x_0, y_0) is a **relative min** if $\Big(\Delta(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0\Big)$ OR $\Big(\Delta(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) > 0\Big)$

    (x_0, y_0) is a **saddle point** if $\Delta(x_0, y_0) < 0$

    The test is **inconclusive** if $\Delta(x_0, y_0) = 0$.

    So, apply the **”First Principles” definitions** of rel min, rel max, and saddle point. (see above)

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EX 11.7.3: Let $h(x,y) = x^6 + y^8$. Find & classify all critical points (CP's) of h.

STEP 1: Compute the 1^{st}-order partials of h:

$$\begin{cases} h_x = \frac{\partial}{\partial x}\left[x^6 + y^8\right] = 6x^5 \\ h_y = \frac{\partial}{\partial y}\left[x^6 + y^8\right] = 8y^7 \end{cases}$$

STEP 2: Set the 1^{st}-order partials equal to zero & solve the resulting **nonlinear** system:

$$\begin{cases} h_x \stackrel{set}{=} 0 \\ h_y \stackrel{set}{=} 0 \end{cases} \Longrightarrow \begin{cases} 6x^5 = 0 \\ 8y^7 = 0 \end{cases} \Longrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \Longrightarrow \boxed{\text{The only CP of } h \text{ is } (0,0)}$$

STEP 3: Compute the 2^{nd}-order partials of h:

$$h_{xx} = \frac{\partial}{\partial x}\Big[h_x\Big] = \frac{\partial}{\partial x}\Big[6x^5\Big] = 30x^4$$

$$h_{yy} = \frac{\partial}{\partial y}\Big[h_y\Big] = \frac{\partial}{\partial y}\Big[8y^7\Big] = 56y^6$$

$$h_{xy} = (h_x)_y = \frac{\partial}{\partial y}\Big[h_x\Big] = \frac{\partial}{\partial y}\Big[6x^5\Big] = 0$$

STEP 4: Compute the **discriminant**, $\Delta(x,y)$:

$$\Delta(x,y) = \det\begin{bmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{bmatrix} = h_{xx}h_{yy} - (h_{xy})^2 = \left(30x^4\right)\left(56y^6\right) - (0)^2 = (30)(56)x^4y^6$$

Now, the value of the discriminant at the CP $(0,0)$ is $\Delta(0,0) = (30)(56)(0)^5(0)^6 = 0$

Since the disciminant at the CP is **zero**, the 2^{nd}-Order Partials Test is **inconclusive**!! **:(**

Therefore, further analysis is needed by appealing to the "First Principles" definitions of rel max, rel min & saddle point:

Observe that $h(x,y) = 0$ only at $(x,y) = (0,0)$.

Elsewhere, $h(x,y) > 0$ since h is a **sum** of **even powers** of x & y.

Let $\mathbb{D}(0,0;r)$ be an **open disk** centered at CP $(0,0)$ with radius $r > 0$.

Then, $h(x,y) \geq h(0,0) \quad \forall (x,y) \in \mathbb{D}(0,0;r)$

Therefore, by the "First Principles" definition, $\boxed{(0,0) \text{ is a } \textbf{relative minimum}.}$

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