DOUBLE INTEGRALS: POLAR COORDINATES [SST 12.3]

EX 12.3.8: Using double integral(s), find the volume of the solid E bounded by the paraboloid $z = 4 - x^2 - y^2$ and xy-plane.

For clarity, let $f(x, y) = 4 - x^2 - y^2$

Find where paraboloid $z = f(x, y)$ intersects the xy-plane ($z = 0$):

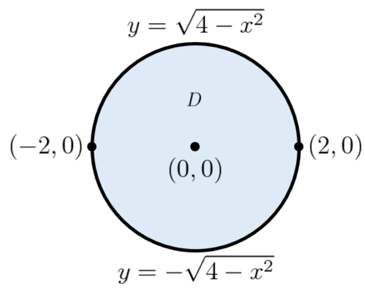
$f(x, y) = 0 \implies 4 - x^2 - y^2 = 0 \implies x^2 + y^2 = 4 \leftarrow$ circle of radius 2 centered at $(0, 0)$

Hence, the circle $x^2 + y^2 = 4$ is the only BC of region D

Notice that $f(x, y) > 0$ **inside** this circle: [e.g. $f(-1, 1) = 4 - (-1)^2 - (1)^2 = 2 > 0$]

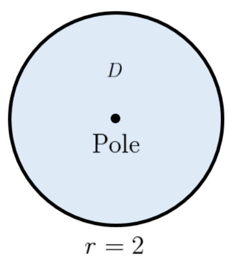
Thus, region D (which is the bottom face of solid E) is the closed disk of radius 2: $x^2 + y^2 \leq 4$

Sketch region D, label the BC & BP's (in rectangular coordinates):



$$\therefore \quad \text{Volume}(E) = \iint_D f \, dA = \underbrace{\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4 - x^2 - y^2\right) \, dy \, dx}_{\text{Too tedious to compute!}}$$

Instead, sketch D, label the **pole** & the BC in polar coordinates:



$$\begin{aligned}
\therefore \quad \text{Volume}(E) &= \iint_D f \, dA \\
&\overset{POLAR}{=} \iint_D f(r\cos\theta, r\sin\theta) \, r \, dr \, d\theta \\
&= \int_{\text{Smallest } \theta\text{-value in } D}^{\text{Largest } \theta\text{-value in } D} \int_{\text{Pole}}^{\text{Outer BC of } D} \left(4 - r^2\right) \, r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^2 \left(4r - r^3\right) \, dr \, d\theta \\
&= \int_0^{2\pi} \left[2r^2 - \frac{1}{4}r^4\right]_{r=0}^{r=2} d\theta \\
&\overset{FTC}{=} \int_0^{2\pi} \left[\left(2(2)^2 - \frac{1}{4}(2)^4\right) - \left(2(0)^2 - \frac{1}{4}(0)^4\right)\right] \, d\theta \\
&= \int_0^{2\pi} 4 \, d\theta \\
&= \Big[4\theta\Big]_{\theta=0}^{\theta=2\pi} \\
&\overset{FTC}{=} [4(2\pi) - 4(0)] = \boxed{8\pi}
\end{aligned}$$